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► **To cite this version:**

Jean-Marc Ginoux, Jaume Llibre. Flow Curvature Method applied to Canard Explosion. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2011, 44 (46), pp.465203. 10.1088/1751-8113/44/46/465203 . hal-01056936

**HAL Id: hal-01056936**

**<https://univ-tln.hal.science/hal-01056936>**

Submitted on 21 Aug 2014

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# FLOW CURVATURE METHOD APPLIED TO CANARD EXPLOSION

JEAN-MARC GINOUX<sup>1</sup> AND JAUME LLIBRE<sup>2</sup>

ABSTRACT. The aim of this work is to establish that the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* can be found according to the *Flow Curvature Method*. This result will be then exemplified with the classical Van der Pol oscillator.

## 1. INTRODUCTION

The classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [30] and Levinson [23] stated that *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. These manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds*<sup>1</sup>. Then, Fenichel [11, 12, 13, 14] theory<sup>2</sup> for the *persistence of normally hyperbolic invariant manifolds* enabled to establish the *local invariance* of *slow invariant manifolds* that possess both expanding and contracting directions and which were labeled *slow invariant manifolds*.

During the last century, various methods have been developed to compute the *slow invariant manifold* or, at least an asymptotic expansion in power of  $\varepsilon$ .

The seminal works of Wasow [32], Cole [6], O'Malley [25, 26] and Fenichel [11, 12, 13, 14] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Method*. According to this theory, existence as well as local invariance of the *slow invariant manifold* of *singularly perturbed systems* has been stated. Then, the determination of the *slow invariant manifold* equation turned into a regular perturbation problem in which one generally expected the asymptotic validity of such expansion to breakdown [26].

Recently a new approach of  $n$ -dimensional singularly perturbed dynamical systems of ordinary differential equations with two time scales, called *Flow Curvature Method* has been developed [17]. In dimension two and three, it consists in considering the *trajectory curves* integral of such systems as *plane* or *space* curves. Based on the use of local metrics properties of *curvature* (*first curvature*) and *torsion* (*second curvature*) resulting from the *Differential Geometry*, this method which does not require the use of asymptotic expansions, states that the location of the points

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*Key words and phrases.* Geometric Singular Perturbation Method, Flow Curvature Method, singularly perturbed dynamical systems, canard solutions.

<sup>1</sup>In other articles the *slow manifold* is the approximation of order  $O(\varepsilon)$  of the *slow invariant manifold*.

<sup>2</sup>The theory of invariant manifolds for an ordinary differential equation is based on the work of Hirsch, *et al.* [19]

where the local *curvature* (resp. *torsion*) of *trajectory curves* of such systems, vanishes, directly provides an approximation of the *slow invariant manifold* associated with two-dimensional (resp. three-dimensional) *singularly perturbed systems* up to suitable order  $O(\varepsilon^2)$  (resp.  $O(\varepsilon^3)$ ). This method gives an implicit non intrinsic equation, because it depends on the euclidean metric.

Solutions of “canard” type have been discovered by a group of French mathematicians [2] in the beginning of the eighties while they were studying relaxation oscillations in the classical Van der Pol’s equation (with a constant forcing term) [31]. They observed, within a small range of the control parameter, a fast transition for the amplitude of the limit cycle varying suddenly from small amplitude to a large amplitude. Due to the fact that the shape of the limit cycle in the phase plane looks as a duck they called it “canard cycle”. Hence, they named this new phenomenon “canard explosion<sup>3</sup>” and triggered a “duck-hunting”.

Many methods have been developed to analyze “canard” solution such as non-standard analysis [2, 8], matched asymptotic expansions [10], or the blow-up technique [9, 22, 28] which extends the *Geometric Singular Perturbation Method* [11, 12, 13, 14].

Meanwhile, two other geometric approaches have been proposed. The first, elaborated by [4] involves *inflection curves*, while the second makes use of the *curvature* of the trajectory curve, integral of any  $n$ -dimensional singularly perturbed dynamical system [16, 17]. This latter, entitled *Flow Curvature Method* will be used in this work in order to compute the bifurcation parameter value leading to a *canard explosion*. Moreover, the total correspondence between the results obtained in this paper for two-dimensional singularly perturbed dynamical systems such as Van der Pol oscillator and those previously established by [2] will enable to highlight another link between the *Flow Curvature Method* and the *Geometric Singular Perturbation Method*.

## 2. SINGULARLY PERTURBED SYSTEMS

According to Tikhonov [30], Takens [29], Jones [20] and Kaper [21] *singularly perturbed systems* may be defined such as:

$$(1) \quad \begin{aligned} \vec{x}' &= \vec{f}(\vec{x}, \vec{z}, \varepsilon), \\ \vec{z}' &= \varepsilon \vec{g}(\vec{x}, \vec{z}, \varepsilon). \end{aligned}$$

where  $\vec{x} \in \mathbb{R}^m$ ,  $\vec{z} \in \mathbb{R}^p$ ,  $\varepsilon \in \mathbb{R}^+$ , and the prime denotes differentiation with respect to the independent variable  $t$ . The functions  $\vec{f}$  and  $\vec{g}$  are assumed to be  $C^\infty$  functions<sup>4</sup> of  $\vec{x}$ ,  $\vec{z}$  and  $\varepsilon$  in  $U \times I$ , where  $U$  is an open subset of  $\mathbb{R}^m \times \mathbb{R}^p$  and  $I$  is an open interval containing  $\varepsilon = 0$ .

In the case when  $0 < \varepsilon \ll 1$ , i.e., is a small positive number, the variable  $\vec{x}$  is called *fast* variable, and  $\vec{z}$  is called *slow* variable. Using Landau’s notation:  $O(\varepsilon^k)$  represents a function  $f$  of  $x$  and  $\varepsilon$  such that  $f(u, \varepsilon)/\varepsilon^k$  is bounded for positive  $\varepsilon$  going to zero, uniformly for  $u$  in the given domain.

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<sup>3</sup>According to Krupa and Szmolyan [22, p. 312] this terminology has been introduced in chemical and biological literature by Brøns and Bar-Eli [3, p. 8707] to denote a sudden change of amplitude and period of oscillations under a very small range of control parameter.

<sup>4</sup>In certain applications these functions will be supposed to be  $C^r$ ,  $r \geq 1$ .

It is used to consider that generally  $\vec{x}$  evolves at an  $O(1)$  rate; while  $\vec{z}$  evolves at an  $O(\varepsilon)$  *slow* rate. Reformulating system (1) in terms of the rescaled variable  $\tau = \varepsilon t$ , we obtain

$$(2) \quad \begin{aligned} \varepsilon \dot{\vec{x}} &= \vec{f}(\vec{x}, \vec{z}, \varepsilon), \\ \dot{\vec{z}} &= \vec{g}(\vec{x}, \vec{z}, \varepsilon). \end{aligned}$$

The dot represents the derivative with respect to the new independent variable  $\tau$ .

The independent variables  $t$  and  $\tau$  are referred to the *fast* and *slow* times, respectively, and (1) and (2) are called the *fast* and *slow* systems, respectively. These systems are equivalent whenever  $\varepsilon \neq 0$ , and they are labeled *singular perturbation problems* when  $0 < \varepsilon \ll 1$ . The label “singular” stems in part from the discontinuous limiting behavior in system (1) as  $\varepsilon \rightarrow 0$ .

In such case system (2) leads to a differential-algebraic system called *reduced slow system* whose dimension decreases from  $m + p = n$  to  $p$ . Then, the *slow* variable  $\vec{z} \in \mathbb{R}^p$  partially evolves in the submanifold  $M_0$  called the *critical manifold*<sup>5</sup> and defined by

$$(3) \quad M_0 := \left\{ (\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = \vec{0} \right\}.$$

When  $D_x f$  is invertible, thanks to implicit function theorem,  $M_0$  is given by the graph of a  $C^\infty$  function  $\vec{x} = \vec{F}_0(\vec{z})$  for  $\vec{z} \in D$ , where  $D \subseteq \mathbb{R}^p$  is a compact, simply connected domain and the boundary of  $D$  is an  $(p - 1)$ -dimensional  $C^\infty$  submanifold<sup>6</sup>.

According to Fenichel theory [11, 12, 13, 14] if  $0 < \varepsilon \ll 1$  is sufficiently small, then there exists a function  $\vec{F}(\vec{z}, \varepsilon)$  defined on  $D$  such that the manifold

$$(4) \quad M_\varepsilon := \left\{ (\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon) \right\},$$

is locally invariant under the flow of system (1). Moreover, there exist perturbed local stable (or attracting)  $M_a$  and unstable (or repelling)  $M_r$  branches of the *slow invariant manifold*  $M_\varepsilon$ . Thus, normal hyperbolicity of  $M_\varepsilon$  is lost via a saddle-node bifurcation of the *reduced slow system* (2).

**Definition 1.** A “*canard*” is a solution of a singularly perturbed dynamical system following the attracting branch  $M_a$  of the slow invariant manifold, passing near a bifurcation point located on the fold of the critical manifold, and then following the repelling branch  $M_r$  of the slow invariant manifold during a considerable amount of time.

Geometrically a *maximal canard* corresponds to the intersection of the attracting and repelling branches  $M_a \cap M_r$  of the slow manifold in the vicinity of a non-hyperbolic point. Canards are a special class of solution of singularly perturbed dynamical systems for which normal hyperbolicity is lost.

<sup>5</sup>It corresponds to the approximation of the slow invariant manifold, with an error of  $O(\varepsilon)$ .

<sup>6</sup>The set  $D$  is overflowing invariant with respect to (2) when  $\varepsilon = 0$ .

### 3. GEOMETRIC SINGULAR PERTURBATION METHOD

Earliest geometric approaches to *singularly perturbed dynamical systems* have been developed by Cole [6], O'Malley [25, 26], Fenichel [11, 12, 13, 14] for the determination of the *slow manifold* equation.

*Geometric Singular Perturbation Method* is based on the following assumptions and theorem stated by Nils Fenichel in the middle of the seventies<sup>7</sup>.

#### 3.1. Assumptions.

- (H<sub>1</sub>) Functions  $\vec{f}$  and  $\vec{g}$  are  $C^\infty$  functions in  $U \times I$ , where  $U$  is an open subset of  $\mathbb{R}^m \times \mathbb{R}^p$  and  $I$  is an open interval containing  $\varepsilon = 0$ .
- (H<sub>2</sub>) There exists a set  $M_0$  that is contained in  $\{(\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = 0\}$  such that  $M_0$  is a compact manifold with boundary and  $M_0$  is given by the graph of a  $C^1$  function  $\vec{x} = \vec{F}_0(\vec{z})$  for  $\vec{z} \in D$ , where  $D \subseteq \mathbb{R}^p$  is a compact, simply connected domain and the boundary of  $D$  is an  $(p - 1)$ -dimensional  $C^\infty$  submanifold. Finally, the set  $D$  is overflowing invariant with respect to (2) when  $\varepsilon = 0$ .
- (H<sub>3</sub>)  $M_0$  is normally hyperbolic relative to the *reduced fast system* and in particular it is required for all points  $\vec{p} \in M_0$ , that there are  $k$  (resp.  $l$ ) eigenvalues of  $D_{\vec{x}}\vec{f}(\vec{p}, 0)$  with positive (resp. negative) real parts bounded away from zero, where  $k + l = m$ .

**Theorem 2** (Fenichel's Persistence Theorem). *Let system (1) satisfying the conditions (H<sub>1</sub>) – (H<sub>3</sub>). If  $\varepsilon > 0$  is sufficiently small, then there exists a function  $\vec{F}(\vec{z}, \varepsilon)$  defined on  $D$  such that the manifold  $M_\varepsilon = \{(\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon)\}$  is locally invariant under (1). Moreover,  $\vec{F}(\vec{z}, \varepsilon)$  is  $C^r$ , and  $M_\varepsilon$  is  $C^r$   $O(\varepsilon)$  close to  $M_0$ . In addition, there exist perturbed local stable and unstable manifolds of  $M_\varepsilon$ . They are unions of invariant families of stable and unstable fibers of dimensions  $l$  and  $k$ , respectively, and they are  $C^r$   $O(\varepsilon)$  close to their counterparts.*

*Proof.* See [11], [20] and [21]. □

**3.2. Invariance.** Generally, Fenichel theory enables to turn the problem for explicitly finding functions  $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$  whose graphs are locally *slow invariant manifolds*  $M_\varepsilon$  of system (1) into regular perturbation problem. Invariance of the manifold  $M_\varepsilon$  implies that  $\vec{F}(\vec{z}, \varepsilon)$  satisfies:

$$(5) \quad \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon) \vec{g}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) = \vec{f}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right).$$

Then, plugging the perturbation expansion:

$$\vec{F}(\vec{z}, \varepsilon) = \sum_{i=0}^{N-1} \vec{F}_i(\vec{z}) \varepsilon^i + O(\varepsilon^N)$$

into (5) enables to solve order by order for  $\vec{F}(\vec{z}, \varepsilon)$ .

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<sup>7</sup>For an introduction to Geometric Singular Perturbation Method see [21].

Taylor series expansion for  $\vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)$  up to terms of order two in  $\varepsilon$  leads at order  $\varepsilon^0$  to

$$(6) \quad \vec{f}(\vec{F}_0(\vec{z}), \vec{z}, 0) = \vec{0}$$

which defines  $\vec{F}_0(\vec{z})$  due to the invertibility of  $D_{\vec{x}}\vec{f}$  and the *Implicit Function Theorem*.

At order  $\varepsilon^1$  we have:

$$(7) \quad D_z \vec{F}_0 \vec{g}(\vec{F}_0, \vec{z}, 0) = D_{\vec{x}} \vec{f}(\vec{F}_0, \vec{z}, 0) \vec{F}_1 + \frac{\partial \vec{f}}{\partial \varepsilon}(\vec{F}_0, \vec{z}, 0),$$

which yields  $\vec{F}_1(\vec{z})$  and so forth.

$$(8) \quad D_{\vec{x}} \vec{f}(\vec{F}_0, \vec{z}, 0) \vec{F}_1 = D_z \vec{F}_0 \vec{g}(\vec{F}_0, \vec{z}, 0) - \frac{\partial \vec{f}}{\partial \varepsilon}(\vec{F}_0, \vec{z}, 0).$$

So, regular perturbation theory enables to build locally *slow invariant manifolds*  $M_\varepsilon$ . But for high-dimensional *singularly perturbed systems slow invariant manifold* asymptotic equation determination leads to tedious calculations.

*Proof.* For application of this technique see [14]. □

**3.3. Slow invariant manifold and canards.** A manifold of canards is an invariant manifold, where first approximation is  $M_0$ . For two-dimensional singularly perturbed dynamical systems with just one fast variable ( $x$ ) and one slow variable ( $y$ ), canards are non generic according to Krupa and Szmolyan [22] and *maximal canards* can only occur in such systems only for discrete values of a control parameter  $\mu$ . It means that in dimension two a one parameter family of singularly perturbed systems is needed to exhibit canard phenomenon. Because along a canard, the differential  $D_x f$  is not always invertible, we can not write the manifold of canards as  $x = F(y, \varepsilon)$ . Thus, we will suppose that  $D_y f$  is invertible and we will try to compute the canard as  $y = F(x, \mu, \varepsilon)$ . See [5] for a theory of this identification of formal series. We consider the following *singularly perturbed dynamical system*:

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y, \mu, \varepsilon), \\ \dot{y} &= g(x, y, \mu, \varepsilon), \end{aligned}$$

with  $x, y \in \mathbb{R}$ , i.e.  $(m, p) = (1, 1)$  and we suppose that due to the nature of the problem perturbation expansions of the canard and of the canard value read:

$$y = F(x, \varepsilon) = \sum_{i=0}^{N-1} F_i(x) \varepsilon^i + O(\varepsilon^N) \quad \text{and} \quad \mu(\varepsilon) = \sum_{i=0}^{N-1} \mu_i \varepsilon^i + O(\varepsilon^N).$$

According to Eq. (5) invariance of the manifold  $M_\varepsilon$  reads:

$$(9) \quad \left( \frac{\partial F}{\partial x}(x, \varepsilon) \right) f(x, F(x, \varepsilon), \mu(\varepsilon), \varepsilon) = \varepsilon g(x, F(x, \varepsilon), \mu(\varepsilon), \varepsilon).$$

To avoid technical complications in the computations below, we assume that, at order  $O(\varepsilon^0)$ , the critical manifold does not depend on the parameter  $\mu$ .

Indeed,

$$\frac{\partial f}{\partial \mu}(x, F_0(x), \mu_0, 0) = 0$$

Then, solving equation (9) order by order provides at:

**Order  $\varepsilon^0$**

$$(10) \quad \frac{\partial F_0}{\partial x}(x)f(x, F_0(x), \mu_0, 0) = 0 \quad \Leftrightarrow \quad f(x, F_0(x), \mu_0, 0) = 0.$$

because the function  $\frac{\partial F_0}{\partial x}(x)$  is almost everywhere non zero. Indeed, the function  $F_0$  is given by the implicit function theorem. In what follows  $f$ ,  $g$ , and their derivatives are evaluated at  $(x, F_0(x), \mu_0, 0)$ , and  $F_0$ ,  $F_1$  and  $F_2$  are evaluated at  $x$ .

**Order  $\varepsilon^1$**

$$F_0' \left( \frac{\partial f}{\partial y} F_1 + \frac{\partial f}{\partial \mu} \mu_1 + \frac{\partial f}{\partial \varepsilon} \right) + F_1' f = g.$$

Since according to what has been stated before, we have:

$$(11) \quad F_1 = \frac{\frac{g}{F_0'} - \frac{\partial f}{\partial \varepsilon}}{\frac{\partial f}{\partial y}}$$

A priori, this function is singular at the bifurcation point  $x_0$  of the fast system, because  $F_0'$  vanishes at this point. To avoid this singularity in function  $F_1$ , the relation  $g(x_0, F(x_0), \mu_0, 0) = 0$  is needed. With an appropriate hypothesis on  $\frac{\partial g}{\partial \mu}$ , it gives a value for  $\mu_0$ .

**Higher order** The computation can be done with the same arguments. When condition of order  $k$  are studied, we have to fix  $F_k$ , and to avoid singularity in  $F_k$  we have to fix  $\mu_{k-1}$ . An example will be done in the next paragraph.

#### 3.4. Van der Pol's "canards". Van der Pol system

$$(12) \quad \begin{aligned} \varepsilon \dot{x} &= f(x, y) = x + y - \frac{x^3}{3}, \\ \dot{y} &= g(x, y) = \mu - x, \end{aligned}$$

satisfies Fenichel's assumptions  $(H_1)$ – $(H_3)$  except on the points  $(x, y) = \pm(1, -\frac{2}{3})$ . The critical manifold is the cubic  $y = x^3/3 - x$ . Thus, the problem is to find a function  $y = F(x, \varepsilon)$  whose graph is locally the *slow invariant manifold*  $M_\varepsilon$  of the Van der Pol system. We write:

$$F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O(\varepsilon^3)$$

and

$$\mu = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + O(\varepsilon^3).$$

The identification we have to perform is

$$\sum_{i=0} \frac{\partial F_i}{\partial x} \varepsilon^i \left( \sum_{i=0} F_i \varepsilon^i - \left( \frac{x^3}{3} - x \right) \right) = \sum_{i=0} \mu_i \varepsilon^{i+1} - \varepsilon x$$

Then, solving order by order provides at:

**Order  $\varepsilon^0$**

$$F_0(x) = \frac{x^3}{3} - x.$$

**Order  $\varepsilon^1$**

$$(13) \quad F_1(x) = -\frac{x - \mu_0}{x^2 - 1}$$

This function is singular at the fold point  $x_0 = 1$  corresponding to the Hopf bifurcation point of the fast system<sup>8</sup>. So, to avoid this singularity in function  $F_1(x)$  we pose:  $\mu_0 = 1$  and thus we have:  $F_1(x) = \frac{-1}{1+x}$ .

**Order  $\varepsilon^2$**

$$(14) \quad F_2(x) = \frac{\mu_1 + \frac{(x - \mu_0)(x^2 + 1 - 2\mu_0 x)}{(x^2 - 1)^3}}{x^2 - 1}$$

Taking into account that  $\mu_0 = 1$  and, in order to avoid singularity in  $F_2(x)$ , we find that  $\mu_1 = -\frac{1}{8}$  and so  $F_2(x) = -\frac{x^2 + 4x + 7}{8(1+x)^4}$ .

**Order  $\varepsilon^3$**

Using the same process, a tedious computation (or, better a computation with the help of a computer) leads to  $\mu_2 = -\frac{3}{32}$ ,  $\mu_3 = -\frac{173}{1024}$ . Thus, the bifurcation parameter value leading to canard solutions reads:

$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3)$$

Then, for  $\varepsilon = 0.01$  one finds again the value obtained numerically by Benoît *et al.* [2, p.99] and Diener [8]:

$$\mu = 0.99874$$

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<sup>8</sup>Due to the symmetry of the vector field:  $(-x, -y, -\mu) \rightarrow (x, y, \mu)$  the same computation could have been done on the fold point  $x_0 = -1$  in the vicinity of which a ‘‘canard explosion’’ also takes place.



The phenomenon of “canard explosion” of Van der Pol system (12) with  $\varepsilon = 0.01$  is exemplified on Fig. 1. where the periodic solution has been plotted in red, the critical manifold in black and the positive fixed point in green. Double arrows indicate the *fast* motion while simple arrows indicate the *slow* motion. Exponentially small variations of the parameter value  $\mu = 0.99874$  enable to exhibit the transition from relaxation oscillation (a) to small amplitude limit cycles (b) via canard cycles (c). Then, at the parameter value  $\mu = 1$  corresponding to the Hopf bifurcation, the canard disappears (d).

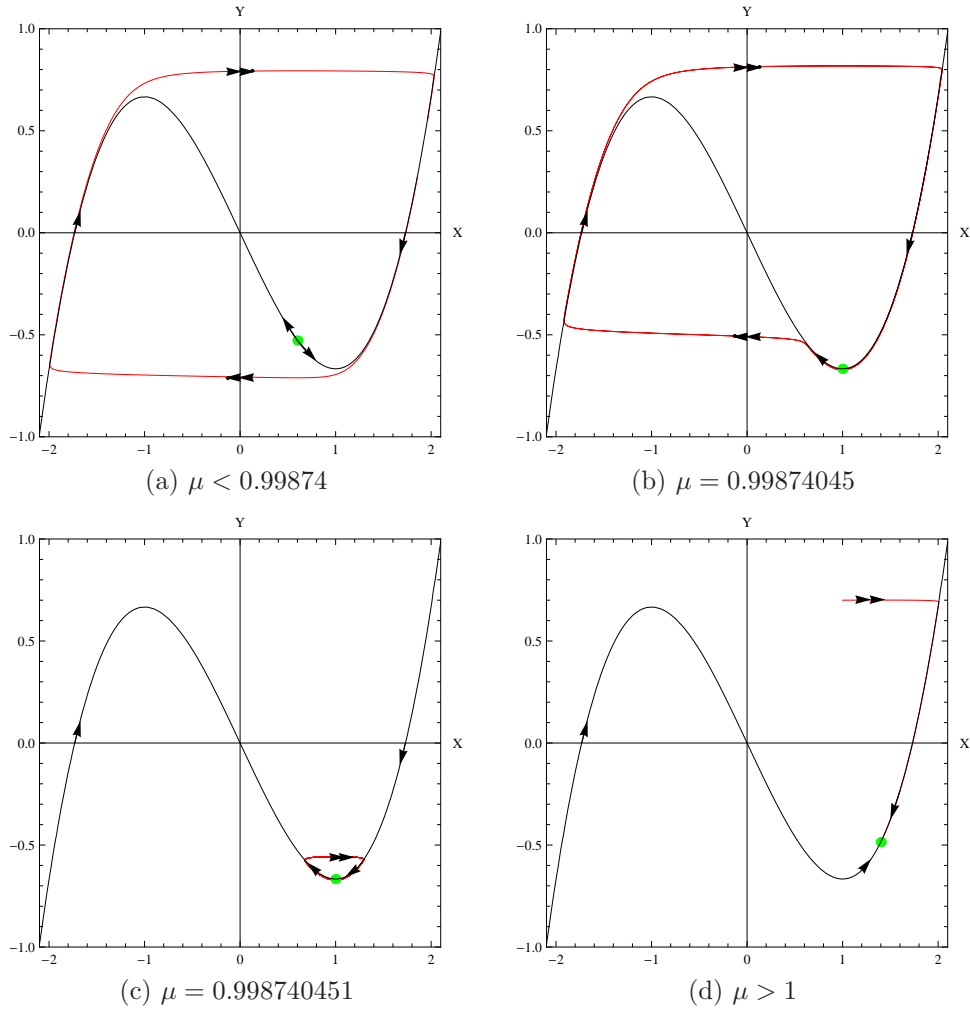


FIGURE 1. Transition from relaxation oscillation to canard explosion.

## 4. FLOW CURVATURE METHOD

Recently, a new approach called *Flow Curvature Method* and based on the use of *Differential Geometry* properties of *curvatures* has been developed, see [16, 17]. According to this method, the highest *curvature of the flow*, i.e. the  $(n-1)^{th}$  *curvature of trajectory curve* integral of  $n$ -dimensional dynamical system defines a *manifold* associated with this system and called *flow curvature manifold*. In the case of  $n$ -dimensional singularly perturbed dynamical system (1) for which with  $\vec{x} \in \mathbb{R}^1$ ,  $\vec{z} \in \mathbb{R}^{n-1}$ , i.e.  $(m, p) = (1, n-1)$  we have the following result.

**Proposition 3.** *The location of the points where the  $(n-1)^{th}$  curvature of the flow, i.e. the curvature of the trajectory curve  $\vec{X}$ , integral of any  $n$ -dimensional singularly perturbed dynamical system vanishes, represents its  $(n-1)$ -dimensional slow manifold  $M_\varepsilon$  the equation of which reads*

$$(15) \quad \phi(\vec{X}, \varepsilon) = \dot{\vec{X}} \cdot (\ddot{\vec{X}} \wedge \ddot{\vec{X}} \wedge \dots \wedge \overset{(n)}{\vec{X}}) = \det(\dot{\vec{X}}, \ddot{\vec{X}}, \ddot{\vec{X}}, \dots, \overset{(n)}{\vec{X}}) = 0$$

where  $\overset{(n)}{\vec{X}}$  represents the time derivatives up to order  $n$  of  $\vec{X} = (\vec{x}, \vec{z})^t$ .

*Proof.* For proof of this proposition see [17, p. 185 and next] and below.  $\square$

**Remark 4.** *First, let's notice that with the Flow Curvature Method the slow manifold is defined by an implicit equation. Secondly, in the most general case of  $n$ -dimensional singularly perturbed dynamical system (1) for which  $\vec{x} \in \mathbb{R}^m$ ,  $\vec{z} \in \mathbb{R}^p$  the Proposition 3 still holds. In dimension three, the example of a Neuronal Bursting Model (NBM) for which  $(m, p) = (2, 1)$  has already been studied by Ginoux et al. [15]. In this particular case, one of the hypotheses of the Tihonov's theorem is not checked since the fast dynamics of the singular approximation has a periodic solution. Nevertheless, it has been established by Ginoux et al. [15] that the slow manifold can all the same be obtained while using the Flow Curvature Method. According to this method, the slow invariant manifold of a three-dimensional singularly perturbed dynamical system for which  $(m, p) = (1, 2)$  is given by the 2<sup>nd</sup> curvature of the flow, i.e. the torsion. In the case of a Neuronal Bursting Model for which  $(m, p) = (2, 1)$  it has been stated by Ginoux et al. [15] that the slow manifold is then given by the 1<sup>st</sup> curvature of the flow, i.e. the curvature. In such a case, the flow curvature manifold is defined by the location of the points where the three-dimensional pseudovector  $\ddot{\vec{X}} \wedge \ddot{\vec{X}}$  vanishes. This condition leads to a nonlinear system of three equations two of which being linearly independent. These two equations define a curve corresponding to the slow invariant manifold<sup>9</sup>. Thus, one can deduce that for a three-dimensional singularly perturbed dynamical system for which  $(m, p)$  the slow manifold is given by the  $p^{th}$  curvature of the flow.*

**4.1. Invariance.** According to Schlomiuk [27] and Llibre et al. [24] the concept of *invariant manifold* has been originally introduced by Gaston Darboux [7, p. 71] in a memoir entitled: *Sur les équations différentielles algébriques du premier ordre et du premier degré* and can be stated as follows.

<sup>9</sup>See also Gilmore et al. [18]

**Proposition 5.** *The manifold defined by  $\phi(\vec{X}, \varepsilon) = 0$  where  $\phi$  is a  $C^1$  in an open set  $U$ , is invariant with respect to the flow of (1) if there exists a  $C^1$  function denoted by  $\kappa(\vec{X}, \varepsilon)$  and called cofactor which satisfies*

$$(16) \quad L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon),$$

for all  $\vec{X} \in U$ , and with the Lie derivative operator defined as

$$L_{\vec{V}}\phi = \vec{V} \cdot \vec{\nabla}\phi = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \dot{x}_i = \frac{d\phi}{dt}.$$

*Proof.* According to Fenichel's Persistence Theorem (see Th. 2) the slow invariant manifold  $M_\varepsilon$  may be written as an explicit function  $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$ , the invariance of which implies that  $\vec{F}(\vec{z}, \varepsilon)$  satisfies

$$(17) \quad \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon) \vec{g}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) = \vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)$$

We write the slow manifold  $M_\varepsilon$  as an implicit function by posing

$$(18) \quad \phi(\vec{x}, \vec{z}, \varepsilon) = \vec{x} - \vec{F}(\vec{z}, \varepsilon) = \phi(\vec{X}, \varepsilon).$$

According to Darboux invariance theorem  $M_\varepsilon$  is invariant if its Lie derivative reads

$$(19) \quad L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon).$$

Plugging Eq. (18) into the Lie derivative (19) leads to

$$L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \dot{\vec{x}} - D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\dot{\vec{z}} = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon),$$

which may be written according to Eq. (2) as

$$L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \frac{1}{\varepsilon}(\vec{f}(\vec{X}, \varepsilon) - \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\vec{g}(\vec{X}, \varepsilon)) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon).$$

Evaluating this Lie derivative in the location of the points where  $\phi(\vec{X}, \varepsilon) = 0$ , i.e.  $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$  leads to

$$L_{\vec{V}}\phi(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) = \frac{1}{\varepsilon}(\vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) - \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\vec{g}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)) = 0,$$

which is exactly identical to Eq. (18) used by Fenichel.  $\square$

**Remark 6.** *This last equation for the invariance of the manifold  $M_\varepsilon$  may be written in a simpler way which implies that  $\phi(\vec{x}, \vec{z}, \varepsilon)$  satisfies*

$$(20) \quad \frac{d}{dt}[\phi(\vec{x}, \vec{z}, \varepsilon)] = 0,$$

on the solutions of the differential system.

**4.2. Slow invariant manifold.** We consider again the following two-dimensional singularly perturbed dynamical system

$$\begin{aligned}\varepsilon \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y),\end{aligned}$$

and we suppose that due to the nature of the problem perturbation expansion reads

$$y = F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O(\varepsilon^3).$$

According to the *Flow Curvature Method* each function  $\vec{F}_i(\vec{z})$  of this perturbation expansion may be found again starting from the *slow manifold implicit equation* (16) as stated in the next result.

**Proposition 7.** *The functions  $F_i(x)$  of the slow invariant manifold associated with a two-dimensional singularly perturbed dynamical system are given by the following expressions*

$$(21) \quad \begin{aligned} F'_n(x) &= \lim_{\varepsilon \rightarrow 0} \left[ \frac{1}{n!} \frac{\partial^n a_{10}}{\partial \varepsilon^n} \right] \text{ with } n \geq 0, \\ F_n(x) &= \lim_{\varepsilon \rightarrow 0} \left[ \frac{1}{n!} \frac{\partial^{n-1} a_{01}}{\partial \varepsilon^{n-1}} \right] \text{ with } n \geq 1, \end{aligned}$$

where

$$a_{10} = - \left. \frac{\frac{\partial \phi_i}{\partial x}}{\frac{\partial \phi_i}{\partial y}} \right|_{y=F(x, \varepsilon)} \quad \text{and} \quad a_{01} = - \left. \frac{\frac{\partial \phi_i}{\partial \varepsilon}}{\frac{\partial \phi_i}{\partial y}} \right|_{y=F(x, \varepsilon)},$$

and

$$\phi_i(x, y, \varepsilon) = \frac{d^{i-1}}{dt^{i-1}} [\phi_1(x, y, \varepsilon)] \quad \text{with } i = 1, 2, \dots, n,$$

where  $\phi_i(x, y, \varepsilon)$  corresponds to the  $i^{\text{th}}$  order approximation in  $\varepsilon$ .

*Proof.* We have that

$$\phi(\vec{X}, \varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0 \quad \Leftrightarrow \quad \phi(x, y, \varepsilon) = 0.$$

Since, for a two-dimensional singularly perturbed dynamical systems this *slow manifold equation* is defined by the *second order tensor of curvature*, i.e. by a determinant involving the first and second time derivatives of the vector field  $\vec{X}$ , it corresponds to the first order approximation in  $\varepsilon$  of the *slow manifold* obtained with the *Geometric Singular Perturbation Method*. So, we denote it by

$$\phi_1(\vec{X}, \varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0.$$

A *third order tensor of curvature* can be easily given by the time derivative of  $\phi_1(\vec{X}, \varepsilon)$ . We denote it by

$$\phi_2(\vec{X}, \varepsilon) = \dot{\phi}_1(\vec{X}, \varepsilon) = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0.$$

Thus,  $\phi_2(\vec{X}, \varepsilon)$  corresponds to the second order approximation in  $\varepsilon$ . Using the same process, we consider the *slow manifold*  $\phi_i(\vec{X}, \varepsilon)$  which corresponds to the  $i^{\text{th}}$  order approximation in  $\varepsilon$ .

Writing the *total differential* of the *slow manifold* we obtain

$$(22) \quad d\phi_i(x, y, \varepsilon) = \frac{\partial\phi_i}{\partial x}dx + \frac{\partial\phi_i}{\partial y}dy + \frac{\partial\phi_i}{\partial\varepsilon}d\varepsilon = 0.$$

Replacing in Eq. (23)  $dy$  by its *total differential*  $dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial\varepsilon}d\varepsilon$  yields

$$(23) \quad d\phi_i(x, y, \varepsilon) = \left( \frac{\partial\phi_i}{\partial x} + \frac{\partial\phi_i}{\partial y} \frac{\partial F}{\partial x} \right) dx + \left( \frac{\partial\phi_i}{\partial\varepsilon} + \frac{\partial\phi_i}{\partial y} \frac{\partial F}{\partial\varepsilon} \right) d\varepsilon.$$

According to Eq. (21)  $\phi_i(x, y, \varepsilon)$  is *invariant* if and only if  $d\phi_i(x, y, \varepsilon) = 0$ , i.e. if

$$(24) \quad \begin{aligned} \frac{\partial\phi_i}{\partial x} + \frac{\partial\phi_i}{\partial y} \frac{\partial F}{\partial x} = 0 &\Leftrightarrow \frac{\partial F}{\partial x} = -\frac{\frac{\partial\phi_i}{\partial x}}{\frac{\partial\phi_i}{\partial y}}, \\ \frac{\partial\phi_i}{\partial\varepsilon} + \frac{\partial\phi_i}{\partial y} \frac{\partial F}{\partial\varepsilon} = 0 &\Leftrightarrow \frac{\partial F}{\partial\varepsilon} = -\frac{\frac{\partial\phi_i}{\partial\varepsilon}}{\frac{\partial\phi_i}{\partial y}}. \end{aligned}$$

By replacing  $y = F(x, \varepsilon)$  by its expression in both parts of Eq. (25) and by setting

$$a_{10} = -\frac{\frac{\partial\phi_i}{\partial x}}{\frac{\partial\phi_i}{\partial y}} \Bigg|_{y=F(x,\varepsilon)} \quad \text{and} \quad a_{01} = -\frac{\frac{\partial\phi_i}{\partial\varepsilon}}{\frac{\partial\phi_i}{\partial y}} \Bigg|_{y=F(x,\varepsilon)},$$

we have that

$$(25) \quad \begin{aligned} \frac{\partial F(x, \varepsilon)}{\partial x} &= F'_0(x) + \varepsilon F'_1(x) + O(\varepsilon^2) = a_{10}, \\ \frac{\partial F(x, \varepsilon)}{\partial\varepsilon} &= F_1(x) + 2\varepsilon F_2(\varepsilon) + O(\varepsilon^2) = a_{01}. \end{aligned}$$

By using a *recurrence reasoning* it may be easily stated that the functions  $F_i(x)$  of the *slow invariant manifold* associated with a two-dimensional singularly perturbed dynamical system are given by the expressions (21).  $\square$

**4.3. Van der Pol's "canards".** We consider again the Van der Pol system (12). All functions  $F_i(x)$  of the perturbation expansion may be deduced from the *slow manifold* equation defined by (16). But, since the determination of  $F_3(x)$ , i.e. the computation  $\mu_2$  requires a *third order tensor of curvature* we consider  $\phi_3(\vec{X}, \varepsilon)$  the second time derivative of  $\phi_1(\vec{X}, \varepsilon)$  which corresponds to the third order approximation in  $\varepsilon$ .

Thus, we find at

**Order  $\varepsilon^0$**

$$F_0'(x) = \lim_{\varepsilon \rightarrow 0} [a_{10}] = -1 + x^2,$$

from which one deduces that

$$F_0(x) = \frac{x^3}{3} - x + C_0,$$

where the constant  $C_0$  may be chosen in such a way that the *critical manifold* can be found again ( $C_0 = 0$ ).

**Order  $\varepsilon^1$**

$$(26) \quad F_1(x) = \lim_{\varepsilon \rightarrow 0} [a_{01}] = \frac{\mu_0 - x}{x^2 - 1}.$$

Thus, one find again exactly the same functions  $F_1(x)$  as those given by *Geometric Singular Perturbation Method* (14) and of course the same value of the bifurcation parameter  $\mu_0 = 1$ .

**Order  $\varepsilon^2$**

$$(27) \quad F_2(x) = \frac{\mu_1 + \frac{(x - \mu_0)(x^2 + 1 - 2\mu_0x)}{(x^2 - 1)^3}}{x^2 - 1}.$$

Taking into account that  $\mu_0 = 1$  we find again exactly the same functions  $F_2(x)$  as those given by *Geometric Singular Perturbation Method* (15) and of course the same value of the bifurcation parameter  $\mu_1 = -1/8$ .

**Order  $\varepsilon^3$**

A simple and direct computation leads to  $\mu_2 = -\frac{3}{32}$ . Thus, the bifurcation parameter value leading to canard solutions reads

$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3).$$

Then, for  $\varepsilon = 0.01$  one finds again the value obtained by Benoît *et al.* [2, p.99] and Diener [8]

$$\mu = 0.99874\dots$$

A program made with Mathematica and available at: <http://ginoux.univ-tln.fr> enables to compute all order of approximations in  $\varepsilon$  of any two-dimensional singularly perturbed systems.

## 5. CONCLUSION

Thus, the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* has been found again with the *Flow Curvature Method*. This result could be also extended to three-dimensional singularly perturbed dynamical systems such as the 3D-autocatalator in which canard phenomenon occurs.

## ACKNOWLEDGMENTS

Authors would like to thank Professor Eric Benoît and the referees for their fruitful advices.

The second author is supported by the grants MIICIN/FEDER MTM 2008-03437, AGAUR 2009SGR410, and ICREA Academia.

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