SELF-EXCITED OSCILLATIONS : from Poincare to Andronov

Jean-Marc Ginoux

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JEAN-MARC GINOUX[1]

RÉSUMÉ. Till recently, the young Russian mathematician Aleksandr' Andronov was considered by many scientists as the first to have applied the concept of *limit cycle*, introduced almost half a century before by Henri Poincaré, in order to state the existence of self-sustained oscillations.

Consequently, if the discovery of a series of "forgotten lectures" given by Poincaré at the École Supérieure des Postes et Télégraphes (today Sup'Telecom ParisTech) in 1908 proves that he had applied his own concept of *limit cycle* to a problem of Wireless Telegraphy preceding thus Andronov of twenty years, it reopens the discussion of Poincaré's French legacy in Dynamical System Theory or, more precisely in Nonlinear Oscillations Theory.

Poincaré's "forgotten lectures" will be presented in the section B and their reception in France before the First World War and in the 1920's in the French engineers community in section C. Then, a special attention will be paid to the role of Jean-Baptiste Pomey who asked Élie Cartan to solve a problem of sustained oscillations in an electrotechnics oscillator and, to the particular case of Alfred Liénard who proved, under certain conditions, existence and uniqueness of a periodic solution for such an oscillator without having regarded this periodic solution as a "Poincaré's limit cycle".

Starting from 1929, Andronov's note at the *Comptes-Rendus* seems to have become the reference in terms of connection with Poincaré's works. So, the reception of Andronov's results by the French scientific community will be analyzed in section D in considering a selection of works published in France between 1929 to 1943.

In the last section, the fact that Poincaré's lectures have been forgotten as well as the fact that neither Cartan nor Liénard have made any connection with Poincaré's works although they have obviously used some of them, will be discussed. Moreover, the role of Jacques Hadamard in the diffusion in France of the works of the Russian schools and of Poincaré's methods will be also pointed out. Thus, the question of Poincaré's legacy will appear in a new perspective.

1. Poincaré's forgotten lectures on Wireless Telegraphy

On July 4^{th} 1902 Poincaré became Professor of Theoretical Electricity at the École Supérieure des Postes et Télégraphes (today Sup'Télecom) in Paris where he taught until 1910. The director of this school, Édouard Éstaunié (1862-1942), also asked him to give a series of conferences every two years in May-June from 1904 to 1912. He told about Poincaré's first lecture of 1904 :

> « Dès les premiers mots, il apparut que nous allions assister au travail de recherche de cet extraordinaire et génial mathématicien ... À chaque obstacle rencontré, une courte pause marquait l'embarras, puis d'un coup d'épaule, Poincaré semblait défier la fonction gênante ... »

In 1908, Poincaré chose as the subject : wireless telegraphy. The text of his lectures was first published weekly in the journal *La Lumière Électrique* [7] before being edited as a book the year after [8]. In the fifth and last part of these lectures entitled : Télégraphie dirigée : oscillations entretenues (Directive telegraphy : sustained oscillations) Poincaré stated a necessary condition for the establishment of a stable regime of sustained oscillations in the singing arc (a forerunner device of the triode used in Wireless Telegraphy). More precisely, he demonstrated the existence, in the phase plane, of a *stable limit cycle.*

1.1. **The singing arc equation.** Starting from the following diagram (see Fig. 1), Poincaré [7, p. 390] explained that this circuit consists of an Electro Motive Force (E.M.F.) of direct current E, a resistance R and a self-induction, and in parallel, a singing arc and another self-induction L and a capacitor. In order to provide the differential equation modeling the sustained oscillations he calls x the capacitor charge and i the current in the external circuit.

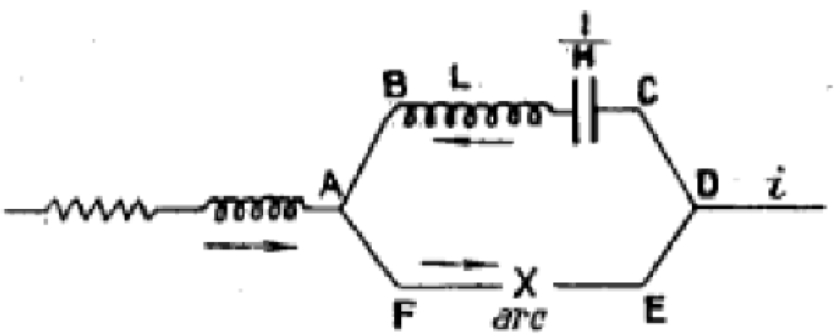

FIGURE 1. Circuit diagram of the singing arc, Poincaré [7, p. 390]

Thus, the current in the branch (ABCD) comprising the capacitor of capacity $1/H$ may be written : $x' = dx/dt$. The current intensity i_a in the branch (AFED) comprising the singing may be written while using Kirchhoff's law : $i_a = i + x'$. Then, Poincaré established the following second order nonlinear differential equation for the sustained oscillations in the singing arc :

$$Lx'' + \rho x' + \theta(x') + Hx = 0 \tag{1}$$

He specified that the term $\rho x'$ corresponds to the internal resistance of the self and various damping while the term $\theta(x')$ represents the E.M.F. of the arc which is related to the intensity by a function, unknown at that time.

1.2. **Stability condition for sustained oscillations and limit cycles.** Then, Poincaré established, twenty years before Andronov [18], that the stability of the periodic solution of the above equation depends on the existence of a closed curve, i.e. of a *stable limit cycle* in the phase plane he has defined in his memoirs ≪ Sur les Courbes définies par une équation différentielle ≫ [**?**, p. 168]. He posed :

$$x' = \frac{dx}{dt} = y\,;\; dt = \frac{dx}{y}\,;\; x'' = \frac{dy}{dt} = \frac{ydy}{dx}$$

Thus, equation (1) becomes :

$$Ly\frac{dy}{dx} + \rho y + \theta\left(y\right) + Hx = 0 \tag{2}$$

Poincaré [7, p. 390] stated then that :

> "Sustained oscillations correspond
> to closed curves if there exist any."

and he gave the following representation for the solution of equation (2) :

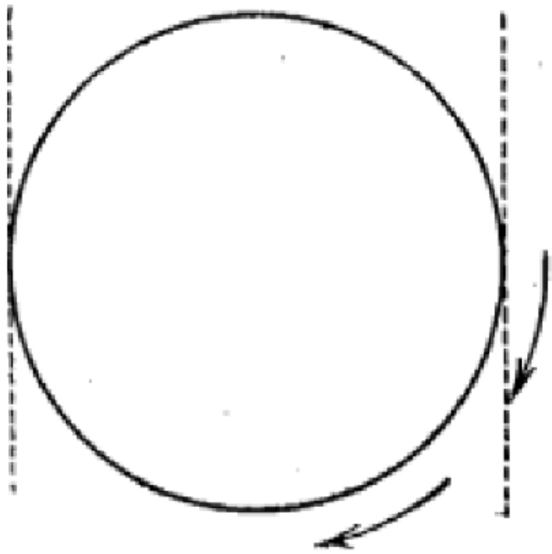

FIGURE 2. Closed curve solution of Eq. (2), Poincaré [1908, p. 390]

Let's notice that this closed curve is only a *metaphor* of the solution since Poincaré does not use any graphical integration method such as *isoclines*.

Then, Poincaré explained that if $y = 0$ then dy/dx is infinite and so, the curve admits vertical tangents. Moreover, if x decreases x' i.e. y is negative. He concluded that the trajectory curves turns in the direction indicated by the arrow (see Fig. 2) and wrote :

> "*Stability condition.* – Let's consider another non-closed curve satisfying the differential equation, it will be a kind of spiral curve approaching indefinitely near the closed curve. If the closed curve represents a stable regime, by following the spiral in the direction of the arrow one should be brought back to the closed curve, and provided that this condition is fulfilled the closed curve will represent a stable regime of sustained waves and will give rise to a solution of this problem."

Then, it clearly appears that the closed curve which represents a stable regime of sustained oscillations is nothing else but a limit cycle as Poincaré [2, p. 261] has introduced it in his own famous memoir "On the curves defined by differential equations" and as Poincaré [3, p. 25] has later defined it in the notice on his own scientific works.

But this, first *giant step* is not sufficient to prove the stability of the oscillating regime. Poincaré had to demonstrate now that the periodic solution of equation (1) (the closed curve) corresponds to a *stable limit cycle*.

1.3. **Possibility condition of the problem : limit cycle stability.** In the following part of his lectures, Poincaré gave what he calls a ≪ condition de possibilité du problème ≫. In fact, he established a stability condition of the periodic solution of equation (1), i.e. a stability condition of the limit cycle under the form of inequality. After multiplying equation (2) by $x'dt$ Poincaré integrated it over one period while taking into account that the first and fourth term vanish since they correspond to the conservative part of this nonlinear equation. He obtained :

$$\rho \int x'^2 dt + \int \theta\left(x'\right) x' dt = 0$$

Then, he explained that since the first term is quadratic, the second one must be negative in order to satisfy this equality. So, he stated that the oscillating regime is stable iff :

$$\int \theta\left(x'\right) x' dt < 0 \tag{3}$$

As exemplified below, Poincaré's approach is identical to which will be used by Alfred Liénard twenty years later.

2. The reception of Poincaré's lectures on Wireless Telegraphy in France

The discovery of these Poincaré's "forgotten lectures" on Wireless Telegraphy implied the analysis of their influence on the French engineers community.

2.1. **Before the First World War (1910-1914).** During this period (1910-1914) one scientific reference to these conferences could be found. It is in the book by Gaston Émile Petit and Léon Bouthillon entitled *La Télégraphie Sans Fil* published in 1910 in which one can read [1] :

> ≪ Le problème de la direction des ondes est donc résoluble. Les ondes peuvent théoriquement être concentrées en faisceau comme les rayons lumineux par des dispositifs appropriés ([1]). ≫
>
> ([1]) Poincaré, Conférences sur la Télégraphie sans fil faites à l'école professionnelle supérieure des Postes et Télégraphes de Paris, 1908, p. 25.

Unfortunately, this unique quotation is very disappointing since it does not refer to the part of Poincaré's lectures concerning the sustained oscillations. Nevertheless, since Petit (ESPT, 1906) and Bouthillon (ESPT, 1907) were students at the École

1. See Petit & Bouthillon [9, p. 128].

Supérieure des Postes et Télégraphes (ESPT) when Poincaré was teaching there, one can suppose that they could have attended his lectures of 1908.

Then, there are several allusions to his lectures in the eulogies made at the time of Poincaré's death on July 17^{th} 1912 and after.

This is the case for example of the French engineer André Blondel (1863-1938) who wrote on July 27^{th} a tribute to Poincaré :

> « De même aussi il avait été conduit à étudier dans ses *Conférences à l'École Supérieure des Postes et Télégraphes* le problème de la propagation de l'électricité, à propos duquel il a développé les recherches de Kohlrausch et poussé plus loin ses résultats. De même il fut amené à s'intéressé à la télégraphie sans fil, qui était pour lui une application des théories qu'il avait développées sur les oscillations électriques [2]. »

The year after, Gaston Darboux (1842-1917) in his historical praise recalled [3] :

> « Les conférences qu'il a données à l'École de Télégraphie nous montrent également combien il se tenait près de l'expérience, et quels services il a rendus aux praticiens.
>
> L'équation, dite des télégraphistes, nous fait connaître, comme on sait, les lois de la propagation d'une perturbation électrique dans un fil. Poincaré intègre cette équation par une méthode générale qui peut s'appliquer à un grand nombre de questions analogues. Le résultat varie suivant la nature du récepteur placé sur la ligne, ce qui se traduit mathématiquement par un changement dans les équations aux limites, mais la même méthode permet de traiter tous les cas.
>
> Dans une seconde série de conférences, Poincaré a étudié le récepteur téléphonique ; un point qu'il a mis particulièrement en évidence, c'est le rôle des courants de Foucault dans la masse de l'aimant.
>
> Enfin, dans une troisième série de conférences, il a traité les diverses questions mathématiques relatives à la télégraphie sans fil : émission, champ en un point éloigné ou rapproché, diffraction, réception, résonance, ondes dirigées, <u>ondes entretenues</u> ([1]). »
>
> ([1]) Ces Conférences ont été publiées dans la collection des cours de l'école et dans la revue *L'Éclairage électrique*.

2.2. **After the First World War (1919-1928).** One might think that his lectures were forgotten because they dealt with an application to a device for Wireless Telegraphy : the singing arc which had become obsolete after the war.

2.2.1. *Janet and the electrical-mechanical analogy (1919).* Of course, during the First World War the triode invented on January 15^{th} 1907 by Lee de Forest (1873-1961) had supplanted the singing arc. However, a French engineer named Paul Janet (1863-1937) published in 1919 a note at the *Comptes-Rendus* in which he established an analogy between the triode and the singing arc. In this paper, entitled « Sur une analogie électrotechnique des oscillations entretenues » and, by using the classical

2. See Blondel [10, p. 100].
3. See Darboux [11, p. 37].

electrical-mechanical analogy, Janet [12] provided the nonlinear differential equation characterizing the oscillations sustained by the singing arc and by the triode :

$$L\frac{d^2i}{dt^2} + \left[R - f'(i)\right]\frac{di}{dt} + \frac{k^2}{K}i = 0 \tag{4}$$

By using the electrical-mechanical analogy and by posing $H \to k^2/K$ and $\theta(x') = f'(i)\frac{di}{dt}$ it is easy to show that Eq. (1) and Eq. (4) are completely analogous. Nevertheless, there is no reference to Poincaré in this note.

2.2.2. *Pomey and the singing arc equation (1920).* On June 28^{th} 1920, a textbook written by the engineer Jean-Baptiste Pomey (1861-1943) entitled : " Introduction à la théorie des courants téléphoniques et de la radiotélégraphie " was published in France. A former student of the École Supérieure des Postes et Télégraphes, Pomey (ESPT, 1883) became Professor of Theoretical Electricity in this school alongside Henri Poincare and then director from 1924 to 1926. In Chapter XIX of his book, devoted to the generation of sustained oscillations Pomey [13, p. 375] wrote :

> ≪ Pour que des oscillations soient engendrées spontanément et s'entretiennent, il ne suffit pas que l'on ait un mouvement périodique, il faut encore que ce mouvement soit stable. ≫

Then, he provided the nonlinear differential equation of the singing arc :

$$Lx'' + Rx' + \frac{1}{C}x = E_0 + ax' - bx'^3 \tag{5}$$

By posing $H = 1/C$, $\rho = R$ and $\theta(x') = -E_0 - ax' + bx'^3$ it is obvious that Eq. (1) and Eq. (5) are completely identical [4]. Moreover, it is striking to observe that Pomey has used exactly the same variable x' as Poincaré to represent the current intensity. Here again, there is no reference to Poincaré. This is very surprising since Pomey was present during the last lecture of Poincaré at the École Supérieure des Postes et Télégraphes in 1912 whose he had written the introduction. So, one can imagine that he could have attended the lecture of 1908.

2.2.3. *Élie and Henri Cartan and the existence of a periodic solution (1925).* On September 28^{th} 1925, Pomey wrote a letter to the mathematician Élie Cartan (1869-1951) in which he asked him to provide a condition for which the oscillations of an electrotechnics device analogous to the singing arc and to the triode whose equation is exactly that of Janet (see Eq. (4)) are sustained. Within ten days, Élie Cartan and his son Henri sent an article entitled : ≪ Note sur la génération des oscillations entretenues ≫ in which they proved the existence of a periodic solution for Janet's equation (4). In fact, their proof is based on a diagram (see Fig. 3) which corresponds exactly to a "first return map" diagram introduced by Poincaré in his memoir ≪ Sur les Courbes définies par une équation différentielle ≫ [2, p. 251]. They wrote :

> ≪ Les points H_1, H_2, H_3 où la courbe coupe la bissectrice correspondent à des solutions *périodiques* (oscillations entretenues) *dont l'existence est ainsi démontrée.* Elles sont périodiques parce qu'en partant d'un minimum donné $-i_1$, le maximum suivant est égal

4. For more details see Ginoux & Lozi [36].

à i_1, par suite le minimum suivant e à $-i_1$, etc. On peut maintenant voir facilement que *toute solution tend vers une solution périodique*[5]. ≫

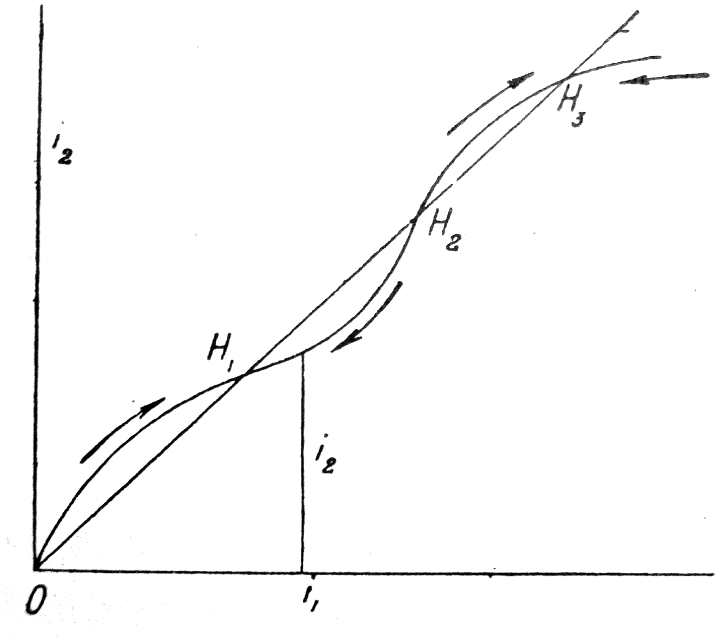

FIGURE 3. "First return map" diagram, Cartan [1925, p. 1199]

Obviously, Fig. 3 exhibits an application of Poincaré's method although there is no reference to his works. Moreover, Cartan didn't recognize in this periodic solution a limit cycle of Poincaré.

2.2.4. *Liénard and the existence and uniqueness of a periodic solution (1928).* Three years after, on May 1928, the engineer Alfred Liénard (1869-1958) published an article entitled ≪ Étude des oscillations entretenues ≫ in which he studied the solution of the following nonlinear differential equation :

$$\frac{d^2x}{dt^2} + \omega f(x)\frac{dx}{dt} + \omega^2 x = 0 \tag{6}$$

Such an equation is a generalization of the well-known Van der Pol's equation and of course of Janet's equation (4). Under certain assumptions on the function $F(x) = \int_0^x f(x)\,dx$ less restrictive than those chosen by Cartan [14] and Van der Pol [15], Liénard [17] proved the existence and uniqueness of a periodic solution of Eq. (6). Then, Liénard [17, p. 906] plotted this solution (see Fig. 4) and wrote :

> ≪ On se rend ainsi compte que la courbe intégrale décrit une sorte de spirale tendant asymptotiquement vers la courbe fermée D. Pour les courbes intégrales extérieures à la courbe fermée, c'est OA_2 qui

5. See Cartan [14, p. 1199].

devient inférieur à OA_1. La courbe se rapproche encore de la courbe D, mais par l'extérieur. En raison de ce fait que toutes les courbes intégrales, intérieures ou extérieures, parcourues dans le sens des temps croissants, tendent asymptotiquement vers la courbe D on dit que le mouvement périodique correspondant est un mouvement *stable.* ≫

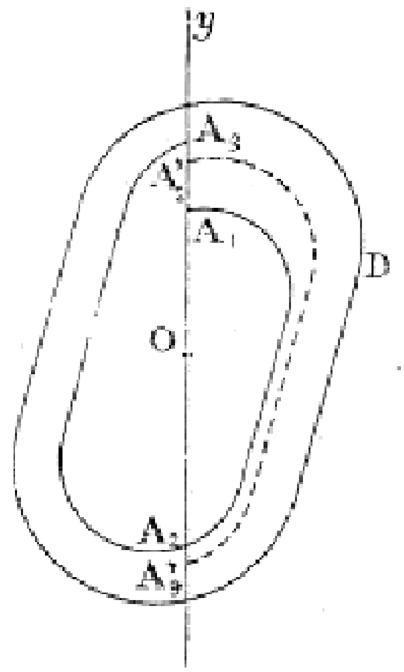

FIGURE 4. Closed curve solution of Eq. (6), Liénard [1928, p. 905].

The Liénard [17, p. 906] explained that the condition for which the "periodic motion" is stable is given by the following inequality :

$$\int_{\Gamma} F(x)\, dy > 0 \tag{7}$$

By considering that the trajectory curve describes the closed curve clockwise in the case of Poincaré and counter clockwise in the case of Liénard, it is easy to show that both conditions (3) and (7) are completely identical [6] and represents an analogue of what is now called "orbital stability". Again, one can find no reference to Poincaré in Liénard's paper. Moreover, it is very surprising to observe that he didn't used the terminology "limit cycle" to describe its periodic solution. The case of Liénard is really a riddle that we will discussed below.

6. For more details see Ginoux [35] and Ginoux [37].

3. The reception of Andronov's results in France

After the Poincaré's death in 1912, the French mathematician Jacques Hadamard (1865-1963) succeeded him at the Academy of Sciences. According to Maz'ya et Shaposhnikova [33, p. 181] Hadamard was elected corresponding member of the Russian Academy of Sciences in 1922 and, foreign member of the Academy of Sciences of the USSR in 1929. That's probably the reason why he was asked to review [7] and to present at the *Comptes-Rendus* on October 14^{th} 1929 a note from the young Russian mathematician Aleksandr' Aleksandrovich Andronov (1901-1952) entitled :

> ≪ Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues. [8] ≫

In this short paper (only three pages), Andronov [18] stated a correspondence between the periodic solution of self-oscillating systems representing for example the oscillations of a radiophysics device used in Wireless Telegraphy and the concept of stable *limit cycle* introduced by Poincaré [2, p. 261].

In order to analyze the reception of Andronov's result, a selection of works published in France and elsewhere during the period (1930-1943) will be briefly presented.

3.1. **Van der Pol's relaxation oscillations and limit cycles (1930).** Looking at the famous plot of the graphical integration of the triode oscillator exhibited by Van der Pol [15, p. 983] (see Fig. 5), one might think that he has immediately recognized that such a periodic solution was a *stable limit cycle* of Poincaré.

This is not the case. It is just after the publication of Andronov's note at the *Comptes-Rendus* that Van der Pol realized that the periodic solution he has plotted (see Fig. 5) was a limit cycle. It was during a lecture given in Paris at the École Supérieure d'Électricité on March 10^{th} and 11^{th} 1930. Van der Pol [20, p. 294] reproduced the figures of his original paper of 1926 (comprising the Fig. 5) and he said :

> ≪ On remarque sur chacune de ces trois figures une courbe intégrale fermée ; c'est un exemple de ce que Henri POINCARÉ a appelé un *cycle limite* ([1]), parce que les courbes intégrales s'en rapprochent asymptotiquement. ≫
>
> ([1]) Voir par exemple : *A. Andronow*, Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues, C. R. **189**, 559 (1929).

Let's notice that although he quotes Poincaré, he makes reference to Andronov. Moreover, he also quotes in the following, the papers of Cartan [14] and that of Liénard [17] but it does not seem that he has ever used their works. During his visits in Paris, Van der Pol was hosted by a young French engineer named Philippe Le Corbeiller (1891-1980) who helped him to translate his talks [9]. Le Corbeiller who was probably present at the École Supérieure d'Électricité on March 10^{th} and 11^{th} 1930 was invited to give a lecture at the *Third International Congress of Applied Mechanics* held in Stockholm from 24^{th} to 29^{th} August 1930. He was accompanied by Van der Pol himself and by Alfred Liénard.

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7. The original of this note is presented in Ginoux [35, p. 494-496].
8. Andronov [18]. It has been pointed out by Ginoux [35, p. 176] that this note has been preceded by a presentation at the Congress of Russian Physicists between 5^{th} and 16^{th} August 1928. See Andronov [16].
9. See Van der Pol [20, p. 312].

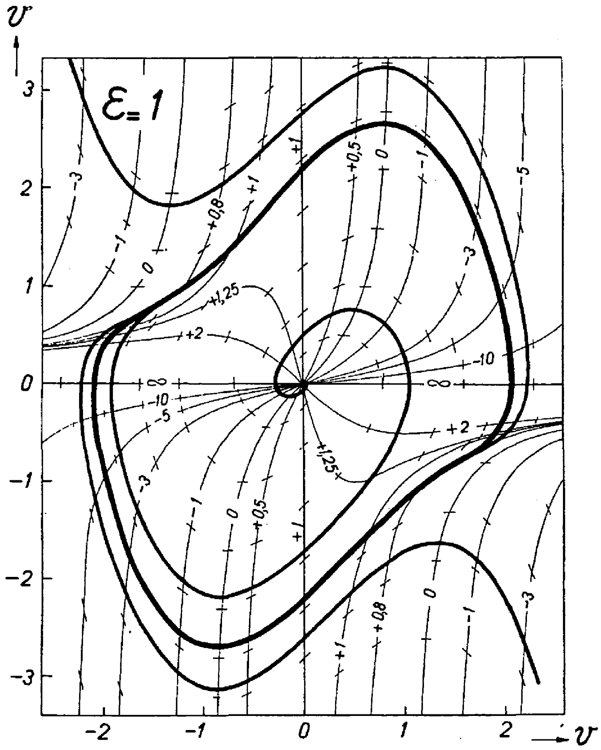


FIGURE 5. Graphical integration of the triode oscillator, Van der Pol [15, p. 983]

3.2. **Le Corbeiller and the Theory of Nonlinear Oscillations (1930).** During his talk entitled ≪ Sur les oscillations des régulateurs ≫ Le Corbeiller [22, p. 211] recalled :

> ≪ Si nous savons que le système machine-régulateur présente effectivement des oscillations périodiques, cela signifiera que parmi les courbes intégrales tracées sur la surface caractéristique il y en a au moins *une* qui est une courbe fermée. Mais le système n'étant plus linéaire à coefficients constants, les solutions infiniment voisines ne seront plus homothétiques à cette courbe, mais s'en approcheront asymptotiquement, c'est-à-dire que la solution périodique correspondra à un *cycle limite* de POINCARÉ, comme l'a fait remarquer M. ANDRONOW. Son amplitude sera ainsi bien déterminée. ≫

Then, he ended his article by this sentence :

> ≪ Je ne puis que renvoyer aux remarquables travaux de cet auteur[10], auxquels M. LIÉNARD et M. ANDRONOW ont apporté des compléments fort intéressants. ≫

As an Historian of Sciences, Le Corbeiller presented during his various lectures a synthesis of the different results obtained in the field of relaxation oscillations.

10. Van der Pol.

Moreover, his contribution for the understanding of the processes of development of the nonlinear oscillations theory is fundamental since he recalls the essential steps that would surely have fallen into oblivion without his intervention. Nevertheless, it is striking to notice that in his famous lecture given in Paris at the Conservatoire National des Arts & Métiers on May 6^{th} and on May 7^{th} 1931, Le Corbeiller [23, p. 4] quoted very few Andronov and never Poincaré :

> « Mais c'est un physicien hollandais, M. Balth. van der Pol, qui, par sa théorie des *oscillations de relaxation* (1926) a fait avancer la question d'une manière décisive. Des savants de divers pays travaillent actuellement à élargir la voie qu'il a tracée ; de ces contributions, la plus importante nous paraît être celle de M. Liénard (1928). Des recherches mathématiques fort intéressantes sont poursuivies par M. Andronow, de Moscou. »

On April 22^{nd} 1932, Le Corbeiller gave a lecture in Paris at the École Supérieure des Postes et Télégraphes where he has been student many years before (ESPT, 1914). Let's recall that Poincaré taught in this school from 1902 till 1912 where he also gave many lectures and in particular his "forgotten lectures" on Wireless Telegraphy.

Discussing the graphical integration of the triode oscillator proposed by Van der Pol [15, p. 983] (See Fig. 5. above), Le Corbeiller [24, p. 708-709] wrote :

> « La théorie générale de ces courbes intégrales fermée, ou *cycles limites*, a été faite par H. Poincaré (2) [1] ; la démonstration de l'existence d'un cycle limite unique, dans ce cas actuel, est due à M. Liénard [16].
>
> (2) D'une manière tout à fait générale, l'équation $\dfrac{dx}{X(x,y)} = \dfrac{dy}{Y(x,y)}$ équivaut aux deux suivantes :
>
> $$\frac{d^2x}{dt^2} - y\left(\frac{Y}{X} + x\right) + x = 0,\ y = \frac{dx}{dt}.$$
>
> Le mémoire cité de Poincaré équivaut donc à l'étude du système oscillant conservatif $\dfrac{d^2x}{dt^2} + x = 0$, soumis à des forces de dissipation et d'entretien dont la résistance est une force *quelconque* de x et de $\dfrac{dx}{dt}$: »
>
> $$F\left(x, \frac{dx}{dt}\right) = y\frac{Y}{X} + x.$$
>
> [1] H. POINCARÉ, Mémoire sur les courbes définies par une équation différentielle, Deuxième partie, *Journal de Mathém. Pures et app.* 8, 251, 1882 ; et *Œuvres*, T. 1, p. 44.
> [16] A. LIÉNARD, Étude des oscillations entretenues, *Rev. gén. d'électr.*, 901 et 946, 1926.

It is very interesting to remark that Le Corbeiller made reference to the original paper of Poincaré and that he also gave many mathematical details on the way of writing the differential equation characterizing the oscillations of a dissipative system.

3.3. **The Liénard riddle (1931).** As recalled above, in his first paper entitled ≪ Étude des oscillations entretenues ≫, Liénard [17] proved the existence and uniqueness of a periodic solution of a generalized Van der Pol's equation without making any connection with Poincaré's works. Then, less than one year after the presentation of Andronov's notes at the *Comptes-Rendus*, Liénard participated with Le Corbeiller and Van der Pol to the *Third International Congress of Applied Mechanics* held in Stockholm from 24^{th} to 29^{th} August 1930 where he presented an article entitled : ≪ Oscillations auto-entretenues ≫. According to the title, one might have thinking that, in this second and last publication on this subject, Liénard would have taken account of Andronov's result and that he would have established a connection between the periodic solution and a Poincaré's limit cycle. But, surprisingly not.

In this work, Liénard [21] first summarized his previous results and then he generalized another result established by Andronov and Witt [19] in their second and last note at the *Comptes-Rendus* in which they studied the "Lyapunov stability" of the periodic solution, *i.e.* the stability of a *limit cycle* or "orbital stability". In order to extend Andronov and Witt's proposition, Liénard [21, p. 176] made use of "variational equations", *i.e.* of a method introduced by Poincaré [2, p. 162] in the first volume of his famous ≪ Méthodes Nouvelles de la Mécanique Céleste ≫ and which corresponds to what is today known under the name of the computation of "characteristics exponents". To do that, he modified his own equation (6) and replaced it by the following which is now know as "Liénard's equation" :

$$\frac{d^2x}{dt^2} + \omega f\left(x, \frac{dx}{dt}\right) + \omega^2 x = 0 \tag{8}$$

Then, he wrote :

> ≪ Si l'équation (8) admet une solution périodique, de période T, la condition pour que cette solution soit stable est que l'intégrale pendant une période de $\dfrac{\partial f\left(x,x'\right)}{\partial x'}dt$ soit positive. La proposition, établie par Messieurs ANDRONOW et WITT ([11]) dans le cas particulier où la fonction $f\left(x,x'\right)$ est très petite se généralise immédiatement. ≫

Thus, Liénard [21] generalizes the result of Andronov and Witt [19] for the stability of a periodic solution, *i.e.* of a limit cycle according to Poincaré's method of "characteristics exponents" but without quoting Poincaré's works and without using the terminology "limit cycle" for describing the stable periodic solution [12]. However, this expression appears in the very first pages of the article of Andronov and Witt [19, p. 256] in footnote :

> ≪ ([2]) Pour la définition des auto-oscillations et la discussion du cas d'un degré de liberté voir A. ANDRONOW, *Les cycle limites de Poincaré et la théorie des oscillations auto-entretenues* (Comptes rendus, 189, 1929, p. 559). ≫

Therefore, it seems very difficult to explain the attitude of Liénard especially since at the first International Conference on Nonlinear Oscillations, to which he

11. Liénard quotes Andronov and Witt [1930].
12. For more details see Ginoux [35, p. 210].

was invited, the question of periodic solutions of type limit cycle has been much discussed.

3.4. **The first "lost" International Conference on Nonlinear Oscillations (1933).** From 28 to 30 January 1933 the first International Conference of Nonlinear Oscillations was held at the Institut Henri Poincaré (Paris) organized at the initiative of the Dutch physicist Balthasar Van der Pol and of the Russian mathematician Nikolaï Dmitrievich Papaleksi. The discovery of this event, of which virtually no trace remains, was made possible thanks to the report written by Papaleksi at his return in USSR. This document has revealed, on the one hand, the list of participants who included French mathematicians : Alfred Liénard, Élie and Henri Cartan, Henri Abraham, Eugène Bloch, Léon Brillouin, Yves Rocard, ... and, on the other hand the content of presentations and discussions. The analysis of the minutes of this conference highlights the role and involvement of the French scientific community in the development of the theory of nonlinear oscillations [13].

According to Papaleksi [25, p. 211], during his talk, Liénard recalled the main results of his study on sustained oscillations :

> "Starting from its graphical method for constructing integral curves of differential equations, he deduced the conditions that must satisfy the nonlinear characteristic of the system in order to have periodic oscillations, that is to say for that the integral curve to be a closed curve, i.e. a limit cycle."

This statement on Liénard must be considered with great caution. Indeed, one must keep in mind that Papaleksi had an excellent understanding of the work of Andronov [18] and that his report was also intended for members of the Academy of the USSR to which he must justified his presence in France at this conference in order to show the important diffusion of the Soviet work in Europe. Despite the presence of MM. Cartan, Lienard, Le Corbeiller and Rocard it does not appear that this conference has generated, for these scientists, a renewed interest in the problem of sustained oscillations and limit cycles. However, although the theory of nonlinear oscillations does not seem to be in France at that time a research priority, it is the subject of several Ph-D theses and monographs discussed below.

3.5. **The French Ph-D theses.** During the period (1936-1943) several Ph-D theses were defended in France on a subject strongly related to nonlinear oscillations. Two of them are briefly recalled below [14].

The first is that of R. Morched-Zadeh [15] who defended a Ph-D thesis at the Faculté des Sciences de l'Université de Toulouse in October 1936 entitled : ≪ Étude des oscillations de relaxation et des différents modes d'oscillations d'un circuit comprenant une lampe néon ≫. In the introduction of his study Morched-Zadeh [27, p. 3] wrote :

> ≪ Au point de vue théorique, les cycles limites de H. POINCARÉ prennent une grande place dans la théorie des oscillations autoentretenues comme l'ont démontré A. ANDRONOW et A. WITT. ≫

13. For more details see Ginoux [35] and Ginoux [37].

14. For instance four Ph-D thesis have been found during this period and completely analyzed by Ginoux [35].

15. No biographic information could be found concerning this student except the fact that he was Iranian but not parent with Lotfi Morched-Zadeh (personal communication).

In this case it is very surprising to observe that Morched-Zadeh made reference to the second and last note of Andronov and Witt [19] at the *Comptes-Rendus* and not to the first which seemed to be better known and most quoted.

The aim of his work is an experimental study of relaxation oscillations of a neon lamp submitted to various oscillating regimes comprising of course the case of self-sustained oscillations. This led him to take the very first pictures of a *limit cycle* on a cathode ray tube oscilloscope (see Fig. 6.)

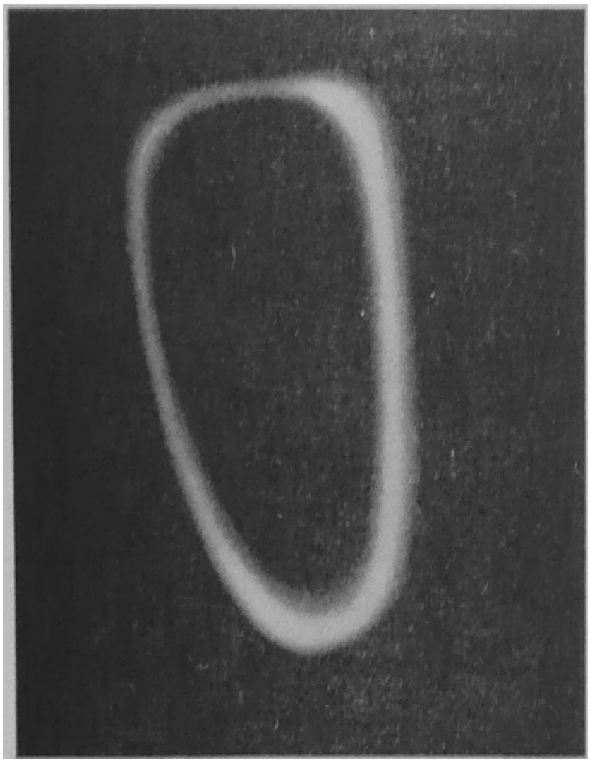


FIGURE 6. Limit cycle of the neon lamp oscillator, Morched-Zadeh [1936, p. 127]

The second is that of Jean Abelé (1886-1961) who was physicist, philosopher and writer[16]. He defended his Ph-D thesis entitled : " Étude d'un système oscillant entretenu à amplitude autostabilisée et application à l'entretien d'un pendule élastique " at the Faculté des Sciences de l'Université de Paris in front of a jury comprising Yves Rocard. In the introduction of his work Abelé [29, p. 18] wrote :

> ≪ À un mouvement périodique *stable* correspond une courbe intégrale fermée *dont s'approchent asymptotiquement en spirales, de l'intérieur et de l'extérieur, pour t croissant, les solutions voisines*. Un des problèmes fondamentaux de la théorie non linéaire consiste dans la recherche de ces courbes fermées, dites *cycles limites*[17]. ≫ [29, p. 18]

16. For biographic details see for example Ginoux [35, p. 325].
17. Abelé quotes Andronov [18].

Although, the definition of a stable limit cycle exactly corresponds to that given by Poincaré himself, Abelé quotes Andronov.

3.6. **Rocard's textbooks.** During the Second World War, the physicist Yves Rocard (1903-1992) published two manuscripts. If the title of the first one ≪ Théorie des Oscillateurs ≫ is very close to that of Andronov and Khaikin [28] published in Russian and entitled : "Теория колебаний [18]" the content is quite different. Rocard [30] proposes a synthesis of several works done in this area as well as a summary of the article of Van der Pol [15] on relaxation oscillations with figures including the Fig. 5 that he commented thus :

> ≪ On voit au fur et à mesure que ε croît, se déformer le cycle limite et apparaître les harmoniques. ≫

This is the single occurrence of the terminology limit cycle in the whole textbook, which is given without any reference [19].

In 1943, Rocard [31] published his ≪ Dynamique Générale des Vibrations ≫ which has been wrongly considered as a textbook on nonlinear oscillations. In fact, in this book which comprises sixteen chapters only three deals with this subject. In chapter XV, Rocard [31, p. 220] recalls the results of Liénard [17] and plots the following figure :

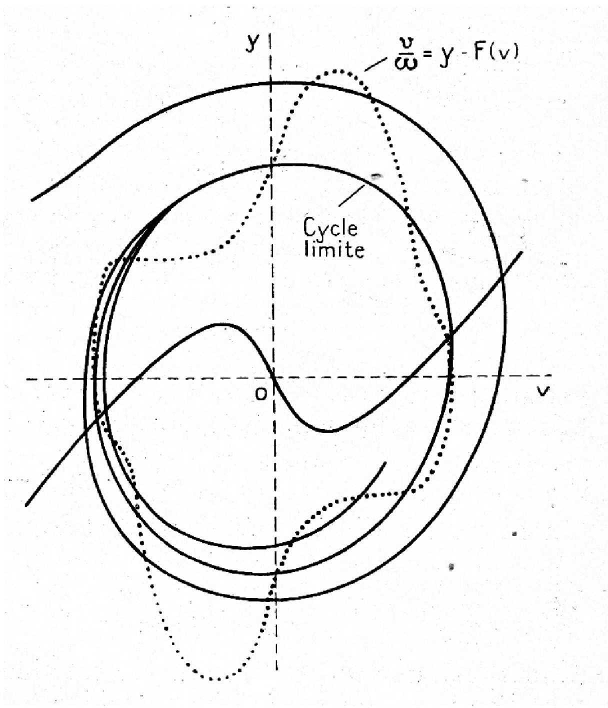


FIGURE 7. Limit cycle of relaxation oscillator, Rocard [31, p. 220]

18. "Theory of oscillations".

19. The question of the absence of references in Rocard's book has been much discussed. Interviewed on this issue, Rocard said that because of the war he was unable to access these documents. In fact it has been shown in Ginoux [35, p. 263] that it was not true.

Then, he explained :

> ≪ . . . on constate (comme autrefois H. Poincaré l'a démontré) que la courbe intégrale s'enroule un certain nombre de fois et tend vers une courbe fermée dite *cycle limite*, qui dans cette représentation correspond au régime d'oscillations permanent. ≫

Here again, there is no reference neither to Poincaré nor to Andronov.

4. Discussion

This study has shown that two major contributions in the development of nonlinear oscillation theory had occurred in France during the first half of the XXth century. The first is the correspondence between the concept of limit cycle and the existence of a stable regime of sustained oscillations in Wireless Telegraphy established by Henri Poincare in 1908 in these "forgotten lectures" at the École Supérieure des Postes et Télégraphes, and the second is the same kind of correspondence, established twenty years later in a more general context by the Russian mathematician Aleksandr' Andronov in his famous note at the *Comptes-Rendus*.

If, the "discovery" of these "forgotten lectures" demonstrated that Poincaré has stated that the periodic solution of a nonlinear differential equation characterizing the nonlinear oscillations of a particular radiophysics device named *singing arc* is nothing else but a stable limit cycle, it has really produced no reaction on the French scientific community.

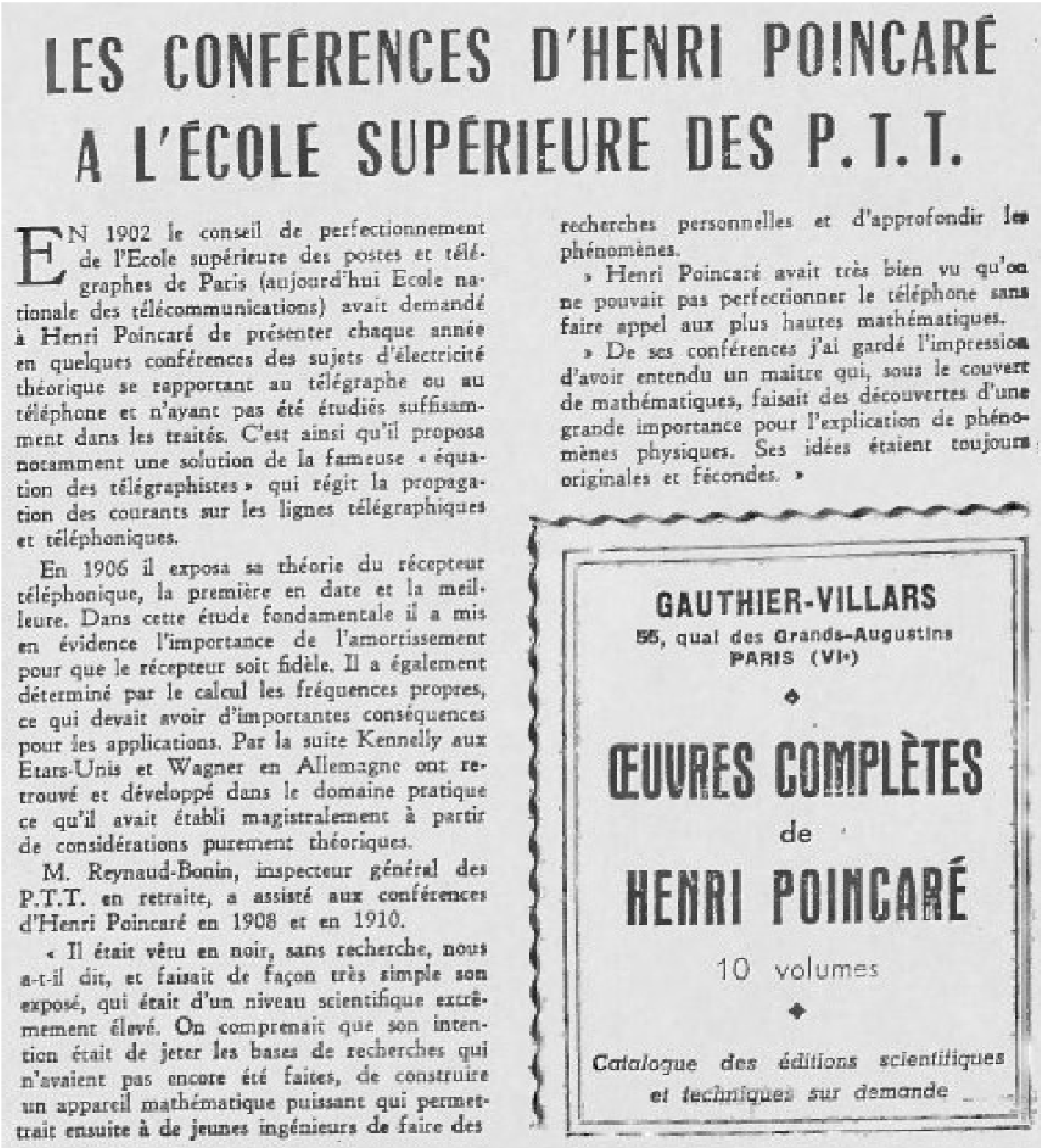
Nevertheless, the analysis of the influence of Andronov's note on this scientific community from 1929 to 1943 has shown that it was nearly the same. Liénard, for unknown reasons didn't make use of the terminology "limit cycle" neither before 1929 nor after. Moreover, although he became probably aware of Andronov's correspondence during the first "lost" International Conference on Nonlinear Oscillations in 1933 he has not pursued his research in this area. But, in 1933, Liénard was 64 years old and near to retirement. This was not the case for Le Corbeiller who has been one of the first to establish a deep connection with Poincaré's works. However, when the Second World War was declared he went to the USA and became a Professor in Harvard. Concerning Rocard, who made only few allusions to Poincaré's concept of limit cycle in his textbooks, he turned to nuclear research immediately after WWII.

Thus, it seems that although France has been a kind of crossroads for the development of nonlinear oscillations theory, nobody has succeeded in unifying French scientists around a research program in this area.

In fact, the mathematician Jacques Hadamard has been deeply involved in this task at various levels. First, he has presented during the 1930's many notes at the French Academy of Sciences on this subject coming from USSR : that of Andronov, of Andronov and Witt but also that of Kryloff and Bogoliouboff. But, he has also presented the works of Poincaré during his seminar at the *Collège de France* from 1919 to 1932. Unfortunately it didn't provide any reaction from the French scientific community.

So, although this community has produced many fundamental results necessary for the development of the nonlinear oscillation theory such as that of Liénard for example there has been no research program like in USSR or in USA and so, no "School of nonlinear" in France. The terrible impact of the WWI and WWII is probably responsible for such a lack of organization.

During the commemoration of the centenary of Poincaré's birth, the newspaper *Le Monde* published on May 15^{th}, 1954 an article entitled ≪ Les conferences de Henri Poincaré à l'École Supérieure des P.T.T. ≫ (See Fig. 8) in which we learn that Eugène Reynaud-Bonin, a former student of this school (ESPT, 1911), has attended to Poincaré's "forgotten lectures" in 1908 and 1910. His interview was reproduced in *Le Monde* (see Fig. 8 below).



LES CONFÉRENCES D'HENRI POINCARÉ A L'ÉCOLE SUPÉRIEURE DES P.T.T.

EN 1902 le conseil de perfectionnement de l'Ecole supérieure des postes et télégraphes de Paris (aujourd'hui Ecole nationale des télécommunications) avait demandé à Henri Poincaré de présenter chaque année en quelques conférences des sujets d'électricité théorique se rapportant au télégraphe ou au téléphone et n'ayant pas été étudiés suffisamment dans les traités. C'est ainsi qu'il proposa notamment une solution de la fameuse « équation des télégraphistes » qui régit la propagation des courants sur les lignes télégraphiques et téléphoniques.

En 1906 il exposa sa théorie du récepteur téléphonique, la première en date et la meilleure. Dans cette étude fondamentale il a mis en évidence l'importance de l'amortissement pour que le récepteur soit fidèle. Il a également déterminé par le calcul les fréquences propres, ce qui devait avoir d'importantes conséquences pour les applications. Par la suite Kennelly aux Etats-Unis et Wagner en Allemagne ont retrouvé et développé dans le domaine pratique ce qu'il avait établi magistralement à partir de considérations purement théoriques.

M. Reynaud-Bonin, inspecteur général des P.T.T. en retraite, a assisté aux conférences d'Henri Poincaré en 1908 et en 1910.

« Il était vêtu en noir, sans recherche, nous a-t-il dit, et faisait de façon très simple son exposé, qui était d'un niveau scientifique extrêmement élevé. On comprenait que son intention était de jeter les bases de recherches qui n'avaient pas encore été faites, de construire un appareil mathématique puissant qui permettrait ensuite à de jeunes ingénieurs de faire des recherches personnelles et d'approfondir les phénomènes.

» Henri Poincaré avait très bien vu qu'on ne pouvait pas perfectionner le téléphone sans faire appel aux plus hautes mathématiques.

» De ses conférences j'ai gardé l'impression d'avoir entendu un maître qui, sous le couvert de mathématiques, faisait des découvertes d'une grande importance pour l'explication de phénomènes physiques. Ses idées étaient toujours originales et fécondes. »

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FIGURE 8. *Le Monde*, May 15^{th}, 1954

Three days later, the physicist Nicolas Minorsky [32] presented a lecture to civil engineers entitled : ≪ Influence d'Henri Poincaré sur l'évolution moderne de la théorie des oscillations non linéaires ≫ in which he began with these words :

> ≪ La répercussion des travaux d'Henri POINCARÉ s'est fait sentir dans presque tous les domaines des sciences appliquées, mais c'est surtout dans la théorie des oscillations qu'elle a provoqué de tels changements que cette théorie est aujourd'hui passablement différente de ce qu'elle soit. ≫

Then, he concluded by this sentence which shows that he didn't have knowledge of Poincaré's "forgotten lectures" :

> ≪ Il est difficile de trouver dans l'histoire de la Science un autre exemple de théorie mathématique développée sans aucune relation aux applications ... qui ait présenté une base aussi parfaite pour l'étude des phénomènes innombrables qui se sont révélés depuis lors, sans qu'il y ait presque rien a changer à cette théorie un demi-siècle plus tard. ≫

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