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Averaging for SDE-BSDE with null recurrent fast component Application to homogenization in a non periodic media

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Abstract

We establish an averaging principle for a family of solutions $(X^\varepsilon, Y^\varepsilon) := (X^{1,\varepsilon}, X^{2,\varepsilon}, Y^\varepsilon)$ of a system of SDE-BSDE with a null recurrent fast component $X^{1,\varepsilon}$. In contrast to the classical periodic case, we can not rely on an invariant probability and the slow forward component $X^{2,\varepsilon}$ cannot be approximated by a diffusion process. On the other hand, we assume that the coefficients admit a limit in a Česaro sense. In such a case, the limit coefficients may have discontinuity. We show that we can approximate the triplet $(X^{1,\varepsilon}, X^{2,\varepsilon}, Y^\varepsilon)$ by a system of SDE-BSDE (X^1, X^2, Y) where $X := (X^1, X^2)$ is a Markov diffusion which is the unique (in law) weak solution of the averaged forward component and Y is the unique solution to the averaged backward component. This is done with a backward component whose generator depends on the variable z . As application, we establish an homogenization result for semilinear PDEs when the coefficients can be neither periodic nor ergodic. We show that the averaged BSDE is related to the averaged PDE via a probabilistic representation of the (unique) Sobolev $W_{d+1,loc}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ -solution of the limit PDEs. Our approach combines PDE methods and probabilistic arguments which are based on stability property and weak convergence of BSDEs in the S-topology.

Keys words: *SDE, BSDEs and PDEs with discontinuous coefficients, weak convergence of SDEs and BSDEs, homogenization, S-topology, Averaging in Česaro sence, Sobolev Spaces, Sobolev solution to semilinear PDEs.*

MSC 2000 subject classifications, 60H20, 60H30, 35J60, 60J35.

1 Introduction

The averaging of stochastic differential equations (SDE) as well as the homogenization of a partial differential equation (PDE) is a process which consists in showing the convergence of the solution of an equation with rapidly varying coefficients towards an equation with simpler (e.g. constant) coefficients.

The two classical situations which were mainly studied are the cases of deterministic periodic and random stationary coefficients. These two situations are based on the existence of an invariant probability measure for some underlying process. The averaged coefficients are then determined as a certain "means" with respect to this invariant probability measure.

There is a vast literature on the homogenization of PDEs with periodic coefficients, see for example the monographs [5, 19, 31] and the references therein. There also exist numerous works on averaging of stochastic differential equations with periodic structures and its connection with homogenization of second order partial differential equations (PDEs). Closer to our concern here, we can quote in particular [7, 8, 9, 12, 18, 20, 28, 33, 34] and the references therein.

In contrast to these two classical situations (deterministic periodic and random stationary coefficients) which were mainly studied, we consider in this paper a different situation, building upon earlier results of [23] and more recently those of [1, 2]. We extend the results of [23] to systems of SDE-BSDEs and those of [1, 2] to the case where the generator f of the BSDE component depends upon the second unknown of the BSDE. As a consequence, we derive an homogenization result for semilinear PDEs when the nonlinear part depends on the solution as well as on its gradient.

In [23], Khasminskii & Krylov consider the averaging of the following family of diffusions process indexed by ε ,

$$\begin{cases} X_s^{1,x,\varepsilon} = x_1 + \int_0^s \varphi\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \\ X_s^{2,x,\varepsilon} = x_2 + \int_0^s \tilde{b}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dr + \int_0^t \tilde{\sigma}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \end{cases} \quad (1.1)$$

where $X^{1,x,\varepsilon}$ is a one-dimensional null-recurrent fast component and $X^{2,x,\varepsilon}$ is a d -dimensional slow component. The function $\varphi = (\varphi_1, \dots, \varphi_k)$ [resp. $\tilde{\sigma} = (\tilde{\sigma}_{ij})_{i,j}$, resp. $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_d)$] is \mathbb{R}^k -valued [resp. $\mathbb{R}^{d \times k}$ -valued, resp. \mathbb{R}^d -valued]. W is a k -dimensional standard Brownian motion. They define the averaged coefficients as limits in the Cesàro sense. With the additional assumption that the presumed limiting SDE has a weakly unique (in law) solution, they prove that the process $(X_s^{1,x,\varepsilon}, X_s^{2,x,\varepsilon})$ converges in distribution towards a Markov diffusion $(X_s^{1,x}, X_s^{2,x})$. As a byproduct, they obtain an homogenization property for the linear PDE associated to $(X_s^{1,x,\varepsilon}, X_s^{2,x,\varepsilon})$ when the limit Cauchy problem, associated to the limit diffusion $(X_s^{1,x}, X_s^{2,x})$, is well posed in the Sobolev space $W_{p,loc}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ for each $p \geq d+2$. Here, $W_{p,loc}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ is the Sobolev space of all functions $u(s, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ such that both u and all the generalized derivatives $D_s u$, $D_x u$, and $D_{xx}^2 u$ belong to $L_{loc}^p(\mathbb{R}_+ \times \mathbb{R}^d)$.

Later, the result of [23] was extended to systems of SDE-BSDE in [1, 2]. Furthermore, in [1, 2] the uniqueness of the averaged SDE-BSDE as well as that of the averaged PDE were established under appropriate conditions, building upon the results from [25]. However, in [1, 2] the backward equation does not depend on the control variable. More precisely, the result of [23] was extended, in [1, 2], to the following SDE-BSDE.

$$\begin{cases} X_s^{1,x,\varepsilon} = x_1 + \int_0^s \varphi\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \\ X_s^{2,x,\varepsilon} = x_2 + \int_0^s \tilde{b}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dr + \int_0^s \tilde{\sigma}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r \\ Y_s^{t,x,\varepsilon} = H(X_t^{x,\varepsilon}) + \int_s^t f\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}, Y_r^{t,x,\varepsilon}\right)dr - \int_s^t Z_r^{t,x,\varepsilon} dM_r^{X^{x,\varepsilon}} \end{cases} \quad (1.2)$$

where $M^{X^{x,\varepsilon}}$ is the martingale part of the process $X^{x,\varepsilon} := (X^{1,x,\varepsilon}, X^{2,x,\varepsilon})$.

The system of SDE-BSDE (1.2) is connected to the semilinear PDE,

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s, x) = (\mathcal{L}^\varepsilon v^\varepsilon)(t, x) + f\left(\frac{x_1}{\varepsilon}, x_2, v^\varepsilon(t, x)\right), & s \geq 0 \\ v^\varepsilon(0, x) = H(x); & x = (x^1, x^2) \in \mathbb{R} \times \mathbb{R}^d. \end{cases} \quad (1.3)$$

where, \mathcal{L}^ε is the infinitesimal generator associated to the Markov process $X^{x,\varepsilon} := (X^{1,x,\varepsilon}, X^{2,x,\varepsilon})$.

In the present paper we consider the situation where the coefficient f depends upon x , y and z . This more general situation will force us to develop a new methodology. That is, the SDE-BSDE in consideration is defined in $[0, t]$ by,

$$\begin{cases} X_s^{1,x,\varepsilon} = x_1 + \int_0^s \varphi\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \\ X_s^{2,x,\varepsilon} = x_2 + \int_0^s \tilde{b}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dr + \int_0^s \tilde{\sigma}\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r \\ Y_s^{t,x,\varepsilon} = H(X_t^{x,\varepsilon}) + \int_s^t f\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}\right)dr - \int_s^t Z_r^{t,x,\varepsilon} dM_r^{X^{x,\varepsilon}} \end{cases} \quad (1.4)$$

where $M^{X^{x,\varepsilon}}$ is the martingale part of the process $X^{x,\varepsilon} := (X^{1,x,\varepsilon}, X^{2,x,\varepsilon})$, i.e.

$$M_s^{X^{x,\varepsilon}} := \int_0^s \sigma\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \quad 0 \leq s \leq t.$$

If we put for $i, j = 1, \dots, d$,

$$b := \begin{pmatrix} 0 \\ \tilde{b} \end{pmatrix}, \quad a_{00} := \frac{1}{2} \sum_{i=1}^k \varphi_i^2, \quad \tilde{\sigma} := (\tilde{\sigma})_{ij}, \quad \sigma := \begin{pmatrix} \varphi \\ \tilde{\sigma} \end{pmatrix}, \quad \tilde{a} := \frac{1}{2}(\tilde{\sigma}\tilde{\sigma}^*), \quad a := \frac{1}{2}(\sigma\sigma^*)$$

(note that a is a $(d+1) \times (d+1)$ matrix, whose rows and columns are indexed from $i = 1$ to $i = d$, while \tilde{a} is a $d \times d$ matrix), and $X^{x,\varepsilon} := \begin{pmatrix} X^{1,x,\varepsilon} \\ X^{2,x,\varepsilon} \end{pmatrix}$, then the SDE-BSDE (1.4) can be rewritten in the form

$$\begin{cases} X_s^{x,\varepsilon} = x + \int_0^s b\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dr + \int_0^s \sigma\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}\right)dW_r, \\ Y_s^{t,x,\varepsilon} = H(X_t^{x,\varepsilon}) + \int_s^t f\left(\frac{X_r^{1,x,\varepsilon}}{\varepsilon}, X_r^{2,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}\right)dr - \int_s^t Z_r^{t,x,\varepsilon} dM_r^{X^{x,\varepsilon}} \end{cases} \quad (1.5)$$

In this case, the nonlinear part of the PDE associated to the SDE-BSDE (1.5) depends on both the solution and its gradient. More precisely, this PDE takes the form

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(t, x) = (\mathcal{L}^\varepsilon v^\varepsilon)(s, x) + f\left(\frac{x_1}{\varepsilon}, x_2, v^\varepsilon(s, x), \nabla_x v^\varepsilon(s, x)\right), \\ v^\varepsilon(0, x) = H(x), \end{cases} \quad (1.6)$$

where \mathcal{L}^ε is the infinitesimal generator associated to the Markov process $X^{x,\varepsilon} := (X^{1,x,\varepsilon}, X^{2,x,\varepsilon})$ which is more precisely defined by

$$\mathcal{L}^\varepsilon := a_{00}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial^2}{\partial^2 x_1} + \sum_{j=1}^d a_{0j}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial^2}{\partial x_1 \partial x_{2j}} + \sum_{i,j=1}^d a_{ij}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial^2}{\partial x_{2i} \partial x_{2j}} + \sum_{i=1}^d b_i^{(1)}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial}{\partial x_{2i}},$$

φ , $\tilde{\sigma}$ and \tilde{b} are the coefficients which were defined above, f and H are real valued measurable functions respectively defined on $\mathbb{R}^{d+1} \times \mathbb{R} \times \mathbb{R}^{d+1}$ and \mathbb{R}^{d+1} .

We want to study the asymptotic behavior of the SDE-BSDE (1.5) when $\varepsilon \rightarrow 0$. Note that under suitable conditions upon the coefficients, the function $\{v^\varepsilon(t, x) := Y_0^\varepsilon, t \geq 0, x = (x_1, x_2) \in \mathbb{R}^{d+1}\}$

solves the PDE (1.6), see e. g. Remark 2.6 in [32]. Therefore, we will also study the asymptotic behavior of the PDE (1.6).

As in [1, 2, 23], we consider here the averaged coefficients as limits in the Cesàro sense. Usually, the averaged coefficients are computed as means with respect to the (unique) invariant probability measure. In our situation, due to the fact that the fast component is null recurrent, we have no invariant probability measure. Therefore the classical methods do not work. Furthermore, since the variable Z^ε enters the generator of the backward component and is not relatively compact in any reasonable topology, the identification of the limit of the finite variation process of the backward component is rather hard to obtain. In particular the methods used in [1, 2] do not work.

In order to prove that the limit problem is well posed, we establish the existence and uniqueness for the limiting SDE-BSDE as well as the unique solvability of the limiting PDE in the Sobolev space $W_{p,loc}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, $p \geq d + 2$. We use Krylov's result [25] and standard arguments of BSDEs to establish the existence and uniqueness of the limiting SDE-BSDE. The unique solvability of the limiting PDE is more difficult to prove. Due to the lack of (Hölder's) regularity of the diffusion coefficient, the pointwise estimates of the gradient can not be obtained in our situation. To avoid these problems, we develop a method which consists in establishing an L^p -local version of the Calderón-Zygmund theorem. Our strategy is based on the $W_{p,loc}^{1,2}$ -estimate for solutions of linear PDE with discontinuous coefficients proved in [14]. We use the Gagliardo-Nirenberg interpolation inequality in order to establish $W_{p,loc}^{1,2}$ -estimates for solution of semilinear PDEs. We then obtain a compactness characterization of a suitable approximating sequence of PDEs from which we derive the existence of solutions in the space $W_{p,loc}^{1,2}$. The uniqueness is then deduced from the uniqueness of the limiting SDE-BSDE and the Itô-Krylov formula.

We now pass to the averaging problem. The lack of a reasonable compactness of (Z^ε) create some difficulties in the identification of the limits. Note also that, since (Z^ε) is not a semimartingale, then the method developed in [1, 2, 23] do not directly apply. To avoid these difficulties, we give an approach which combines PDE methods with probabilistic arguments. Indeed, building on the PDEs, we construct a sequence of semimartingales $(Z^{\varepsilon,n})$ that we substitute to (Z^ε) . This allows us to use the method developed in [1, 2, 23]. Next, we show that the problems with $(Z^{\varepsilon,n})$ and that with (Z^ε) average to the same limit. The limits are obtained by combining a regularization procedure, a stability property and weak convergence techniques already used in [1, 2, 12, 23]. Let also note that, in a periodic media, some authors have studied the asymptotic behavior of the the PDE (1.6). We refer to Gaudron and Pardoux [15] in the particular PDEs whose nonlinearity term depends upon the gradient in a quadratic growth manner. The case where the nonlinearity depends fully upon the gradient have been considered by Delarue [12], who developed some of the methods which are needed in this paper.

The paper is organized as follows: In section 2, we give the formulation of the problem and state the main results. Sections 3 and 4 are devoted to the proofs of the two main theorems.

2 Formulation of the Problem and the main results

2.1 Notations

For a given function $g(x)$, we define, whenever they exist, the following limits

$$g^+(x_2) := \lim_{x_1 \rightarrow +\infty} \frac{1}{x_1} \int_0^{x_1} g(t, x_2) dt, \quad g^-(x_2) := \lim_{x_1 \rightarrow -\infty} \frac{1}{x_1} \int_0^{x_1} g(t, x_2) dt$$

and $g^\pm(x) := g^+(x_2)1_{\{x_1 > 0\}} + g^-(x_2)1_{\{x_1 \leq 0\}}$.

Let $\rho(x) := a_{00}(x)^{-1}$. The assumptions we shall make below will allow us to define the averaged

coefficients \bar{b} , \bar{a} and \bar{f} by:

$$\begin{aligned}\bar{b}_i(x) &:= \frac{(\rho b_i)^\pm(x)}{\rho^\pm(x)}, \quad i = 1, \dots, d \\ \bar{a}_{ij}(x) &:= \frac{(\rho a_{ij})^\pm(x)}{\rho^\pm(x)}, \quad i, j = 0, 1, \dots, d \\ \bar{f}(x, y, z) &:= \frac{(\rho f)^\pm(x, y, z)}{\rho^\pm(x)}.\end{aligned}\tag{2.1}$$

It is worth noting that \bar{b} , \bar{a} and \bar{f} can be discontinuous at $x_1 = 0$.

2.2 Assumptions

The following conditions will be used in this paper.

Assumption (A)

(A1) The functions \tilde{b} , $\tilde{\sigma}$, φ are uniformly Lipschitz in (x) . Moreover, for each x_1 their derivatives in x_2 up to and including second order derivatives are bounded continuous functions of x_2 .

(A2) There exist positive constants λ and C_1 such that for every x and ξ , we have

$$\xi^* a \xi \geq \lambda \|\xi\|^2$$

and

$$\begin{cases} (i) \ a_{00}(x) \leq C_1 \\ (ii) \ \sum_{i=1}^d [\tilde{a}_{ii}(x) + b_i^2(x)] \leq C_1(1 + |x_2|^2) \end{cases}$$

Assumption (B) Limits in the Cesàro sense.

(B1) We assume that, as x_1 tends to $\pm\infty$,

$$\frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) dt \quad (\text{resp. } \frac{1}{x_1} \int_0^{x_1} D_{x_2} \rho(t, x_2) dt, \quad \text{resp. } \frac{1}{x_1} \int_0^{x_1} D_{x_2}^2 \rho(t, x_2) dt) \quad \text{tends to}$$

$$\rho^\pm(x_2) \quad (\text{resp. } D_{x_2} \rho^\pm(x_2), \quad \text{resp. } D_{x_2}^2 \rho^\pm(x_2)) \quad \text{uniformly in } x_2.$$

We refer to $\rho^\pm(x_2)$ as a limit in the Cesàro sense.

Here and below $D_{x_2} g$ and $D_{x_2}^2 g$ respectively denote the gradient vector and the matrix of second derivatives in x_2 of g .

(B2) For $i = 0, \dots, d$, $j = 1, \dots, d$, the coefficients ρb_j , $D_{x_2}(\rho b_j)$, $D_{x_2}^2(\rho b_j)$, $\rho \tilde{a}_{ij}$, $D_{x_2}(\rho \tilde{a}_{ij})$, $D_{x_2}^2(\rho \tilde{a}_{ij})$ have averages in the Cesàro sense.

(B3) For any function $g \in \{\rho, \rho b_j, D_{x_2}(\rho b_j), D_{x_2}^2(\rho b_j), \rho \tilde{a}_{ij}, D_{x_2}(\rho \tilde{a}_{ij}), D_{x_2}^2(\rho \tilde{a}_{ij})\}$, there exists a bounded function α such that

$$\begin{cases} \frac{1}{x_1} \int_0^{x_1} g(t, x_2) dt - g^\pm(x) = (1 + |x_2|^2) \alpha(x), \\ \lim_{|x_1| \rightarrow \infty} \sup_{x_2 \in \mathbb{R}^d} |\alpha(x)| = 0. \end{cases}\tag{2.2}$$

Assumption (C)

(C1) There exist $K > 0$ and $p \in \mathbb{N}^*$ such that for every $(x, y, y', z, z') \in \mathbb{R}^{d+1} \times \mathbb{R}^2 \times \mathbb{R}^{1 \times k} \times \mathbb{R}^{1 \times k}$

$$\left\{ \begin{array}{l} (i) \quad |f(x, y, z) - f(x, y', z')| \leq K(|y - y'| + |z - z'|) \\ (ii) \quad |f(x, y, z)| \leq K(1 + |x_2|^p + |y| + |z|) \\ (iii) \quad |H(x)| \leq K(1 + |x_1|^p + |x_2|^p) \text{ and } H \text{ belongs to } \mathcal{W}_{p,loc}^2(\mathbb{R}^{d+1}) \end{array} \right.$$

(C2) ρf has a limit in the Cesàro sense and there exists a bounded measurable function β such that

$$\left\{ \begin{array}{l} \frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) f(t, x_2, y, z) dt - (\rho f)_\pm(x, y, z) = (1 + |x_2|^2 + |y|^2 + |z|^2) \beta(x, y, z) \\ \lim_{|x_1| \rightarrow \infty} \sup_{(x_2, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times (d+1)}} |\beta(x, y, z)| = 0, \end{array} \right. \quad (2.3)$$

where $(\rho f)^\pm(x, y, z) := (\rho f)^+(x_2, y, z) 1_{\{x_1 > 0\}} + (\rho f)^-(x_2, y, z) 1_{\{x_1 \leq 0\}}$.

(C3) For every x_1 , ρf has derivatives up to second order in x_2, y, z and these derivatives are bounded and satisfy (C2).

(C4) For every x_1 , the derivatives of f in x_2, y and z up to and including second order derivatives are bounded continuous functions.

Assume that **(A)**, **(B)**, **(C)** are satisfied. It is well known that:

For every $\varepsilon > 0$ and every (t, x) , the system of SDE-BSDE (1.5) has a unique solution which we denote by $(X_s^{x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon})_{0 \leq s \leq t}$ such that,

- $(Y^{t,x,\varepsilon}, Z^{t,x,\varepsilon})$ is $\mathcal{F}^{X^{x,\varepsilon}}$ adapted, where $\mathcal{F}^{X^{x,\varepsilon}}$ denotes the filtration generated by the process $X^{x,\varepsilon}$. More precisely, $(X^{x,\varepsilon}, Y^{t,x,\varepsilon}, Z^{t,x,\varepsilon})$ is adapted to the filtration \mathcal{F}^B generated by the Brownian motion B .

- $\sup_\varepsilon \mathbb{E}(\sup_{0 \leq s \leq t} |Y_r^{t,x,\varepsilon}|^2 + \int_0^t |Z_r^{t,x,\varepsilon} \sigma(X_r)|^2 dr) < \infty$.

- For every $\varepsilon > 0$, the semilinear PDE (1.6) has a unique solution v^ε in $\mathcal{C}^{1,2}$.

- Note that, since a is uniformly elliptic, we also have $\sup_\varepsilon \mathbb{E} \int_0^t |Z_r^{t,x,\varepsilon}|^2 dr < \infty$. Moreover, we have the relation

$$v^\varepsilon(t, x) = Y_0^{t,x,\varepsilon}.$$

Let \bar{a}, \bar{b} and \bar{f} be the averaged coefficients defined by (1.6). For a fixed (t, x) , let $(X_s^x, Y_s^{t,x}, Z_s^{t,x})_{s \in [0,t]}$ denote the solution of the following system of SDE-BSDE

$$\left\{ \begin{array}{l} X_s^x = x + \int_0^s \bar{b}(X_r^x) dr + \int_0^s \bar{\sigma}(X_r^x) dW_r, \quad 0 \leq s \leq t. \\ Y_s^x = H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^t Z_r^{t,x} dM_r^{X^x}, \quad 0 \leq s \leq t, \end{array} \right. \quad (2.4)$$

where M^{X^x} is the martingale part of X^x .

The PDE associated to the averaged SDE-BSDE (2.4) is given by

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial s}(s, x) = (\bar{L}v)(s, x) + \bar{f}(x, v(s, x), \nabla_x v(s, x)), \quad s \geq 0. \\ v(0, x) = H(x). \end{array} \right. \quad (2.5)$$

where \bar{L} is the infinitesimal generator associated to the process X^x and given by,

$$\bar{L}(x) := \sum_{i,j} \bar{a}_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \bar{b}_i(x) \frac{\partial}{\partial x_i}, \quad (2.6)$$

Our aim is show that,

- 1) equations (2.4) and (2.5) have (in some sense) unique solutions $(Y_s^{t,x}, Z_s^{t,x})$ and v .
- 2) $(X_s^{x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon})$ converges in law to $(X_s^x, Y_s^{t,x}, Z_s^{t,x})$,
- 3) v^ε converges to v in a topology which will be specified below.

According to Khasminskii and Krylov [23] and Krylov [25], we deduce

Proposition 2.1. *Assume that (A), (B) are satisfied. For each $x \in \mathbb{R}^{d+1}$, the forward component $X^{x,\varepsilon} := (X^{1,x,\varepsilon}, X^{2,x,\varepsilon})$ converges in law to the continuous process $X^x = (X^{1,x}, X^{2,x})$ in $C([0, t]; \mathbb{R}^{d+1})$, equipped with the uniform topology. Moreover, X^x is the unique (in law) weak solution of the forward component of the system of equations (2.4).*

2.3 The main results

Proposition 2.2. *(Uniqueness of the averaged BSDE) Assume (A), (B), (C) be satisfied. Then, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}$, the backward component of the system of equations (2.4) has a unique solution $(Y^{t,x}, Z^{t,x})$ such that,*

- (a) $(Y^{t,x}, Z^{t,x})$ is \mathcal{F}^X -adapted and $(Y_s^{t,x}, \int_s^t Z_r^{t,x} dM_r^{X^x})_{0 \leq s \leq t}$ is continuous.
- (b) $\mathbb{E}(\sup_{0 \leq s \leq t} |Y_s^{t,x}|^2 + \int_0^t |Z_r^{t,x} \sigma(X_r^x)|^2 dr) < \infty$.
- (c) Moreover, $Y_0^{t,x}$ is deterministic.

The uniqueness means that, if (Y^1, Z^1) and (Y^2, Z^2) are two solutions of the backward component of (2.4) satisfying (a)–(b) then, $\mathbb{E}(\sup_{0 \leq s \leq t} |Y_s^1 - Y_s^2|^2 + \int_0^t |Z_r^1 \sigma(X_r) - Z_r^2 \sigma(X_r)|^2 dr) = 0$

Proof. Thanks to Remark 3.5 of [33], it is enough to prove existence and uniqueness of solutions for the BSDE

$$Y_s^{t,x} = H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^t Z_r^{t,x} dW_r, \quad 0 \leq s \leq t.$$

Since f satisfies (C) and ρ is bounded, one can easily verify that \bar{f} is uniformly Lipschitz in (y, z) , i.e. satisfies (C1)(i). Existence and uniqueness of a solution follow from standard results for BSDEs, see e. g. [32]. Finally, since $(Y_s^{t,x})$ is $\mathcal{F}_s^{X^x}$ -adapted then $Y_0^{t,x}$ is measurable with respect to a trivial σ -algebra and hence it is deterministic. \blacksquare

The following theorem is closely related to the previous proposition. It shows that the averaged PDE is uniquely solved. It will also be used in the averaging of the SDE-BSDE as well as in the averaging of the PDE. However, this theorem is interesting in its own since it establishes existence, uniqueness and $W_{p,\text{loc}}^{1,2}([0, t] \times \mathbb{R}^d)$ -regularity (for any $p \geq d+2$) of the solution for semilinear PDEs with discontinuous coefficients. It extends, in some sense, the result of [14] to semilinear PDEs.

Theorem 2.3. *Assume that (A), (B), (C) are satisfied. Then, equation (2.5) has a unique solution v such that $v \in W_{p,\text{loc}}^{1,2}([0, t] \times \mathbb{R}^d)$ for any $p \geq d+2$. Moreover, this solution satisfies $v(t, x) = Y_0^{t,x}$.*

The averaging of the backward component of equation (1.5) is given by the following theorem.

Theorem 2.4. *[Averaging of the SDE-BSDE (1.5)] Assume that (A), (B), (C) hold. Then, the sequence of processes $(Y_s^{t,x,\varepsilon}, \int_s^t Z_r^{t,x,\varepsilon} dM_r^{X^\varepsilon})_{0 \leq s \leq t}$ converges in law to $(Y_s^{t,x}, \int_s^t Z_r^{t,x} dM_r^{X^x})_{0 \leq s \leq t}$ in $D([0, t]; \mathbb{R}^2)$, equipped with the \mathbf{S} -topology. Here M^{X^x} is the martingale part of X^x and $(Y_s^{t,x}, Z_s^{t,x})$ is the unique solution of the backward component of equation (2.4).*

Remark 2.1. In [23], the proof is mainly based on the fact that X^ε is a semimartingale. Similarly, in [1] the semimartingale property which enjoy X^ε and Y^ε plays an essential role, see remark 5.1 in [1]. If we try to follow [23] and [1], we need that Z^ε be a semimartingale also. Unfortunately Z^ε is not a semimartingale. Our strategy then consists in replacing Z^ε by an “approximate” semimartingale. The task is to construct a continuous function v , which is smooth enough such that the process $(v(s, X_s), \nabla_x v(s, X_s)) := (Y_s, Z_s)$ is a unique solution of the limit BSDE. To this end, by a compactness argument, we consider the mollified coefficients $(\bar{a}^n, \bar{b}^n, \bar{f}^n, H^n)$ and the associated solution v^n . Note that since our diffusion coefficient a is discontinuous, then we can not obtain a uniform bound for $\nabla_x v^n$. We show that the sequence (v^n) can be estimated in $\mathcal{W}_{p,loc}^{1,2}$ uniformly in n . We then deduce a compactness characterization of the approximate sequence from which we derive the weak convergence towards the function v . Further, we substitute Z^ε by $\nabla_x v^n(\cdot, X^\varepsilon)$ in the BSDE-equation (2.5).

Corollary 2.5. (Averaging of the PDE (1.6)) Assume (A), (B), (C) hold. Then, for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}$, $v^\varepsilon(t, x) \rightarrow v(t, x)$, as $\varepsilon \rightarrow 0$.

3 Proof of Theorem 2.3

Let $\bar{a}_{ij}^n, \bar{b}_i^n, \bar{f}^n, H^n$ denote a regularizing sequence of $\bar{a}_{ij}, \bar{b}_i, \bar{f}, H$ respectively. For each $n \geq 1$, $\bar{a}_{ij}^n, \bar{b}_i^n, \bar{f}^n, H^n$ are infinitely differentiable bounded functions with bounded derivatives of every order. H^n converges uniformly on compact sets towards H . Moreover $\bar{a}_{ij}^n, \bar{b}_i^n, \bar{f}^n$ converge respectively to $\bar{a}, \bar{b}, \bar{f}$ in L_{loc}^p for every $p > d + 2$. We assume in addition that the assumptions (A1), (A2) and (C1) are satisfied along the sequence, with constants which do not depend upon n .

Let us define

$$\bar{L}^n(x) := \sum_{i,j} \bar{a}_{ij}^n(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \bar{b}_i^n(x) \frac{\partial}{\partial x_i}.$$

Consider the sequence of PDEs on $[0, t] \times \mathbb{R}^{d+1}$,

$$\begin{cases} \frac{\partial v^n}{\partial s}(s, x) = \bar{L}^n(x)v^n(s, x) + \bar{f}^n(x, v^n(s, x), \nabla_x v^n(s, x)) = 0 \\ v^n(0, x) = H^n(x) \end{cases} \quad (3.1)$$

Note that, for each n , the PDE (3.1) admit a unique solution v^n which is twice continuously differentiable in (s, x) and three times continuously differentiable in x , see e.g. [27], Theorem 5.1, p. 320.

Using standard arguments of SDEs and BSDEs one can show that there exists a constant k_1 not depending on n such that, for every (s, x) ,

$$|v^n(s, x)| \leq k_1(1 + |x|^p). \quad (3.2)$$

Moreover for each n , thanks to Theorem 7.1, chapter VII, in Ladyzhenskaya et al. [27], or Proposition 3.3 in Ma et al. [29] (see also the probabilistic approach of Delarue [12] Thm. 6.1, pp. 85-89), there are constants k_n^2 and k_n^3 such that

$$\sup_{(s,x) \in [0,t] \times \mathbb{R}^{d+1}} |\nabla_x v^n(s, x)| \leq k_n^2 \quad \text{and} \quad \sup_{(s,x) \in [0,t] \times \mathbb{R}^{d+1}} |D_{xx}^2 v^n(s, x)| \leq k_n^3 \quad (3.3)$$

3.1 Compactness of the sequence v_n

We now give an a priori L^p -bounds for the derivatives of v_n .

Proposition 3.1. *For every $p \in [1, \infty[$ and $R > 0$ small enough, there exists a positive constant $C(C_1, K, p, R, t, k_1)$ not depending on n , such that*

$$\int_0^t \int_{B(0, R/2)} [|\partial_s v^n|^p + |\nabla_x v^n|^p + |D_{xx}^2 v^n|^p] dx ds \leq C(C_1, K, p, R, t, k_1)$$

Replacing v by $v - H$, the PDE (2.5) is reduced to a similar PDE with a null terminal datum. Therefore, we can and do assume, throughout the proof of Proposition 3.1, that $H = 0$.

To establish this Proposition, we need some preparation and lemmas. We first recall the Gagliardo-Nirenberg interpolation inequality which plays an important role (Theorem 3, sect. 4, Chap. 8 in Krylov [26], see also Theorem 7.28, Chapter VII, in Gilbarg & Trudinger [16]):

Lemma 3.2. *(The Gagliardo-Nirenberg inequality). Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded open set. For any $p \geq 1$, there exists a constant $C = C(p, d, \text{diameter}(\Omega))$ such that for every function $\psi \in W_p^2(\Omega)$,*

$$\|\nabla_x \psi\|_{L_p(\Omega)} \leq C \left\{ \|\psi\|_{W_p^2(\Omega)} \right\}^{\frac{1}{2}} \left\{ \|\psi\|_{L_p(\Omega)} \right\}^{\frac{1}{2}}. \quad (3.4)$$

It follows from this inequality that, for every $r > 0$ there exists $c = c(p, r, d) > 0$ such that for every $\varepsilon > 0$,

$$\begin{aligned} \int_0^t \int_{B(0, r)} |\nabla_x v^n(s, x)|^p dx ds &\leq \varepsilon \int_0^t \int_{B(0, r)} |D_{xx}^2 v^n(s, x)|^p dx ds \\ &+ c(p, r, d)(1 + \varepsilon^{-1}) \int_0^t \int_{B(0, r)} |v^n(s, x)|^p dx ds \end{aligned} \quad (3.5)$$

Since v^n is uniformly bounded on compact set, then according to the previous inequality and the fact that v^n satisfies the PDE (3.1), it remains to show that for any small enough $r > 0$,

$$\sup_n \int_0^t \int_{B(0, r)} |D_{xx}^2 v^n(t, x)|^p dx dt < \infty \quad (3.6)$$

In order to establish the previous inequality, we use the strategy developed in the proof of Theorem 9.11 in Gilbarg & Trudinger [16]. We rewrite the PDE (3.1) as follows

$$\begin{cases} \frac{\partial v^n}{\partial s}(s, x) = \bar{a}_{ij}^n(x_1, 0) \frac{\partial^2 v^n}{\partial x_i \partial x_j}(s, x) + g_n(s, x) = 0, & s \in (0, t) \\ v^n(0, x) = 0 \end{cases} \quad (3.7)$$

where

$$\begin{aligned} g_n(s, x) &:= [\bar{a}_{ij}^n(x) - \bar{a}_{ij}^n(x_1, 0)] \frac{\partial^2 v^n}{\partial x_i \partial x_j}(s, x) + \bar{b}_i^n(x) \frac{\partial v^n}{\partial x_i}(s, x) \\ &+ \bar{f}^n(x, v^n(s, x), \nabla_x v^n(s, x)) \end{aligned}$$

For $R > 0$ and $s \in [0, t]$, we set

- $Q_{s,t,R} := [s, t] \times B(0, R)$, where $B(0, R)$ denotes the ball of radius R .

- $meas(Q_{s,t,R})$ denotes the Lebesgue measure of the set $Q_{s,t,R}$.

For $\sigma \in (0, 1)$, we put $\sigma' := \frac{(1+\sigma)}{2}$ and consider $\eta \in C_0^\infty(B_R)$ a cut-off function $\eta : \mathbb{R}^{d+1} \rightarrow [0, 1]$ satisfying the following properties,

$$\begin{cases} \eta(x) &= 1, & \text{if } x \in B(0, \sigma R), \\ \eta(x) &= 0, & \text{if } |x| \geq \sigma' R, \\ |\nabla_x \eta(x)| &\leq 4(1-\sigma)^{-1}R^{-1} & \text{if } \sigma R \leq |x| \leq \sigma' R, \\ |D_{xx}^2 \eta(x)| &\leq 16(1-\sigma)^{-2}R^{-2} & \text{if } \sigma R \leq |x| \leq \sigma' R \end{cases}$$

Clearly the function $u^n := \eta v^n$ solves the PDE

$$\begin{cases} \frac{\partial u^n}{\partial s}(s, x) = \bar{a}_{ij}^n(x_1, 0) \frac{\partial^2 u^n}{\partial x_i \partial x_j}(s, x) + G_n(s, x) = 0, & s \in (0, T) \\ u^n(0, x) = 0 \end{cases}$$

where, $G_n(s, x) := v^n \bar{a}_{ij}^n(x_1, 0) \frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2\bar{a}_{ij}^n(x_1, 0) \frac{\partial v^n}{\partial x_i} \frac{\partial \eta}{\partial x_j} + \eta g_n(s, x)$

Since \bar{a}^n is bounded in x_1 and locally Lipschitz with respect to x_2 , uniformly w.r.t. n , \bar{b}^n satisfies (A2) and \bar{f}^n satisfies (C1-ii), we deduce that G_n is bounded on $[0, t] \times \mathbb{R}^{d+1}$. Let D be an arbitrary bounded subset of \mathbb{R}^{d+1} . Since $\bar{a}_{ij}^n(\cdot, 0)$ and G_n are bounded, and G_n has a compact support, then according to Theorem 2.5 from Doyoon & Krylov [14], there exists a positive constant $C = C(d, C_1, K)$ not depending on n such that for every n , we have

$$u^n \in W_p^{1,2}([0, t] \times D) \quad \text{and} \quad \|u^n\|_{W_p^{1,2}([0, t] \times D)} \leq C \|G_n\|_{L^p([0, t] \times D)}. \quad (3.8)$$

From the definition of the function η , we see that

$$\|D_{xx}^2 v^n\|_{L^p(Q_{0,t,\sigma R})} \leq \|D_{xx}^2 u^n\|_{L^p(Q_{0,t,\sigma' R})} \quad (3.9)$$

According to inequalities (3.8) and (3.9), it remains to estimate $\int_0^t \int_{B(0,\sigma' R)} |G_n(s, x_1, x_2)|^p dx ds$.

We have

$$\int_0^t \int_{B(0,\sigma' R)} |G_n(s, x_1, x_2)|^p dx ds \leq A_1 + A_2 + A_3 \quad (3.10)$$

where

$$\begin{aligned} A_1 &:= C(p) \int_0^t \int_{B(0,\sigma' R)} |v^n|^p |\bar{a}_{ij}^n(x_1, 0)|^p |D_{xx}^2 \eta(x)|^p dx ds \\ A_2 &:= C(p) \int_0^t \int_{B(0,\sigma' R)} |\bar{a}_{ij}^n(x_1, 0)|^p |\nabla_x v^n|^p |\nabla_x \eta(x)|^p dx ds \\ A_3 &:= C(p) \int_0^t \int_{B(0,\sigma' R)} |g_n(s, x)|^p dx ds \end{aligned}$$

The following lemma gives estimates for A_1, A_2 and A_3 .

Lemma 3.3. *Let $Q := Q_{0,t,R}$. For every p , there exist a positive constant $C(p)$ such that for every $\varepsilon > 0$,*

$$(i) \quad A_1 \leq C(p)(1-\sigma)^{-2p} R^{-2p} \int_0^t \int_{B(0,\sigma' R)} |v^n|^p dx ds$$

$$\begin{aligned}
(ii) \quad A_2 &\leq C(p)(1-\sigma)^{-p}R^{-p} \left[\varepsilon \int_0^t \int_{B(0, \sigma'R)} |D_{xx}^2 v^n|^p dx ds + (1+\varepsilon^{-1}) \int_0^t \int_{B(0, \sigma'R)} |v^n|^p dx ds \right] \\
(iii) \quad A_3 &\leq C(p) \left\{ TR^{d+1} + R^p + (R^p + \varepsilon) \int_0^t \int_{B(0, \sigma'R)} |D_{xx}^2 v^n(s, x_1, x_2)|^p dx ds \right. \\
&\quad \left. + (1+R^p)(1+\varepsilon^{-1}) \int_0^t \int_{B(0, \sigma'R)} |v^n|^p dx ds \right\}.
\end{aligned}$$

Proof. $C(p)$ denotes a constant which may vary from line to line.

Inequality (i) follows from the properties of η and the boundness of $\bar{a}_{ij}^n(x_1, 0)$.

We use the properties of η , the boundedness of $\bar{a}_{ij}^n(x_1, 0)$ and inequality (3.5) to get inequality (ii).

We now show inequality (iii). We have

$$\int_0^t \int_{B(0, \sigma'R)} |g_n(s, x)|^p dx ds \leq (I_1^n + I_2^n + I_3^n)$$

with

$$\begin{aligned}
I_1^n &:= \int_0^t \int_{B(0, \sigma'R)} |\bar{a}_{ij}^n(x) - \bar{a}_{ij}^n(x_1, 0)|^p \left| \frac{\partial^2 v^n}{\partial x_i \partial x_j}(s, x) \right|^p dx ds \\
I_2^n &:= \int_0^t \int_{B(0, \sigma'R)} |\bar{b}_i^n(x)|^p \left| \frac{\partial v^n}{\partial x_i}(s, x) \right|^p dx ds \\
I_3^n &:= \int_0^t \int_{B(0, \sigma'R)} |\bar{f}^n(x, v^n(s, x), \nabla_x v^n(s, x))|^p dx ds
\end{aligned}$$

Since \bar{a}_{ij}^n is uniformly Lipschitz in x_2 , we obtain

$$I_1^n \leq \sup_Q (|x_2|^p) \int_0^t \int_{B(0, \sigma'R)} |D_{xx}^2 v^n(s, x_1, x_2)|^p dx ds \quad (3.11)$$

Noticing that \bar{b}^n satisfies assumption (A2-ii) then using inequality (3.5), we obtain

$$\begin{aligned}
I_2^n &\leq C_1 (1 + \sup_Q |x_2|^p) \left[\varepsilon \int_0^t \int_{B(0, \sigma'R)} |D_{xx}^2 v^n|^p dx ds \right. \\
&\quad \left. + c_1 (1 + \varepsilon^{-1}) \int_0^t \int_{B(0, \sigma'R)} |v^n|^p dx ds \right] \quad (3.12)
\end{aligned}$$

Thanks to assumption (C) and inequality (3.5) we deduce

$$\begin{aligned}
I_3^n &\leq K \left(meas(Q) + \sup_Q |x_2|^p + \int_0^t \int_{B(0, \sigma'R)} |v^n(s, x_1, x_2)|^p dx ds \right. \\
&\quad \left. + \varepsilon \int_0^t \int_{B(0, \sigma'R)} |D_{xx}^2 v^n(s, x_1, x_2)|^p dx ds + c_1 (1 + \varepsilon^{-1}) \int_0^t \int_{B(0, \sigma'R)} |v^n|^p dx ds \right) \quad (3.13)
\end{aligned}$$

Combining (3.11), (3.12) and (3.13), we deduce the desired result. Lemma 3.3 is proved. \blacksquare

Lemma 3.4. (L^p_{loc} estimate of $D_{xx}^2 v^n$). For every $p \in [1, \infty[$ and $R > 0$ small enough, there exists a positive constant $C' = C'(C_1, k, p, R, t, k_1)$ not depending on n , such that

$$\int_0^t \int_{B(0, R/2)} |D_{xx}^2 v^n|^p dx ds \leq 2R^{-2p} C'$$

Proof. Using inequalities (3.8), (3.9), (3.10) and Lemma 3.3, we show that

$$\begin{aligned} & (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma R)} |D_{xx}^2 v^n(s, x)|^p dx ds \\ & \leq C(p) \left\{ 1 + (1-\sigma)^p R^p (1 + \varepsilon^{-1}) + (1-\sigma)^{2p} R^{2p} [1 + 2(1 + \varepsilon^{-1})] \int_0^t \int_{B(0, \sigma' R)} |v^n(s, x)|^p dx ds \right. \\ & \quad + (1-\sigma)^{2p} R^{2p} \left[\varepsilon (1-\sigma)^{-p} R^{-p} + \sup_Q (|x_2|^p) (1 + \varepsilon) + 2\varepsilon \right] \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n(s, x)|^p dx ds \\ & \quad \left. + K (1-\sigma)^{2p} R^{2p} (meas(Q) + \sup_Q (|x_2|^p)) \right\} \end{aligned}$$

Using inequality (3.2) and the fact that $|x| \leq R$ in the set $Q := Q_{0,t,R}$, we show that there exists a positive constant $C(C_1, K, R, p, k_1, \varepsilon, meas(Q))$ such that

$$\begin{aligned} & (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma R)} |D_{xx}^2 v^n|^p dx ds \\ & \leq C(C_1, K, R, p, k_1, \varepsilon, meas(Q)) \\ & \quad + C(p) (1-\sigma)^{-p} R^{-p} \left[\varepsilon (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \quad + C(p) \sup_Q (|x_2|^p) \left[(1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \quad + C(p) (1 + \sup_Q |x_2|^p) \left[\varepsilon (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \quad + C(p) \left[\varepsilon (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \end{aligned}$$

Let $\bar{\Lambda} := 1 + \sup_Q |x_2|^p$. We Choose $\varepsilon := \frac{1}{4} \{ 2^{2p} \bar{\Lambda} C(p) [(1-\sigma)^{-p} R^{-p} + 2] \}^{-1}$ and R be sufficiently small so that $2^{2p} C(p) \sup_Q (|x_2|^p) \leq \frac{1}{4}$ then use the fact that $\frac{1-\sigma}{2} = 1 - \sigma'$ to obtain

$$\begin{aligned} (1-\sigma)^{2p} R^{2p} \int_0^t \int_{B(0, \sigma R)} |D_{xx}^2 v^n|^p dx ds & \leq \frac{1}{2} \left[(1-\sigma')^{2p} R^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \quad + C(C_1, K, p, R, t, k_1) \end{aligned}$$

Passing to the \sup on σ' and σ , we get

$$\begin{aligned} & R^{2p} \left[\sup_{0 < \sigma < 1} (1-\sigma)^{2p} \int_0^t \int_{B(0, \sigma R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \leq \frac{1}{2} R^{2p} \sup_{0 < \sigma' < 1} \left[(1-\sigma')^{2p} \int_0^t \int_{B(0, \sigma' R)} |D_{xx}^2 v^n|^p dx ds \right] \\ & \quad + C(C_1, K, p, R, t, k_1) \end{aligned}$$

It follows that

$$R^{2p} \left[\sup_{0 < \sigma < 1} (1 - \sigma)^{2p} \int_0^t \int_{B(0, \sigma R)} |D_{xx}^2 v^n|^p dx ds \right] \leq 2C(C_1, K, p, R, t, k_1)$$

The proof is finished by taking $\sigma := 1/2$. ■

Proof of Proposition 3.1. Thanks to inequality (3.2), inequality (3.5) and Lemma 3.4, we deduce that $\sup_n \|\nabla_x v^n\|_{L_p([0, t] \times B(0, R/2))}$ is bounded. Since v^n satisfies the PDE (3.1), we deduce that $\sup_n \|\partial_s v^n\|_{L_p([0, t] \times B(0, R/2))}$ is bounded also. Therefore, there exists a positive constant $C = C(C_1, K, p, R, t, k_1)$ such that

$$\sup_n \int_0^t \int_{B(0, R/2)} [|v^n|^p + |\partial_s v^n|^p + |\nabla_x v^n|^p + |D_{xx}^2 v^n|^p] dx ds \leq C \quad (3.14)$$

Proposition 3.1 is proved. ■

Proof of Theorem 2.3. Inequalities (3.14) and (3.2) express that for every $R > 0$ small enough,

$$\sup_n \|v^n\|_{W_p^{1,2}([0, t] \times B(0, R/2))} \leq C(R, k_1, t, p)$$

Since, any ball $B(0, R')$ can be covered by a finite number of balls of radius $R/2$, and the proof of Proposition 3.1 can be easily adapted to proving the same estimate in a ball of radius $R/2$ centered around any point in \mathbb{R}^{d+1} we deduce that

$$\sup_n \|v^n\|_{W_p^{1,2}(Q_{0,t,R'})} < \infty. \quad (3.15)$$

Therefore v^n converges weakly to v in the space $W_p^{1,2}([0, t] \times Q)$, and v solves the PDE (2.5) *a.e.*

We now prove the uniqueness of solution in $W_{p,loc}^{1,2}$. Let $(X_s^x, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq t}$ be a solution of the FBSDE system

$$X_s^x = x + \int_0^s \bar{b}(X_r^x) dr + \int_0^s \bar{\sigma}(X_r^x) dW_r, \quad 0 \leq s \leq t; \quad (3.16)$$

$$Y_s^{t,x} = H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^t Z_r^{t,x} dM_r^{X^x}, \quad 0 \leq s \leq t. \quad (3.17)$$

For $p \geq d + 2$, take any solution $v \in W_{p,loc}^{1,2}$ of the PDE (2.5). The Itô-Krylov formula shows that the process $(v(t-s, X_s^x), \nabla_x v(t-s, X_s^x), 0 \leq s \leq t)$ is a solution of (3.17). Hence $v(t, x) = Y_0^{t,x} = \mathbb{E}(Y_0^{t,x})$. Since (3.17) has a unique solution, $v(t, x)$ is written as the expectation of a uniquely characterized functional of $(X_s^x)_{0 \leq s \leq t}$. But uniqueness in law holds for (3.16) (see Proposition 2.1), consequently the law of X^x is uniquely characterized, hence the solution v of (2.5) is unique in $W_{p,loc}^{1,2}$. ■

As consequence of Theorem 2.3 and the Sobolev embedding Theorem, we have

Corollary 3.5. v^n converges uniformly to v on any compact subset of $\mathbb{R}_+ \times \mathbb{R}^{d+1}$.

4 Proof of Theorem 2.4.

In order to simplify the notation throughout the proof of Theorem 2.4, we will suppress the superscript x (resp. (t, x)) from the processes $(X^x, Y^{t,x}, Z^{t,x})$ and $(X^{x,\varepsilon}, Y^{t,x,\varepsilon}, Z^{t,x,\varepsilon})$. That is, we will respectively replace $(X^x, Y^{t,x}, Z^{t,x})$ by (X, Y, Z) and $(X^{x,\varepsilon}, Y^{t,x,\varepsilon}, Z^{t,x,\varepsilon})$ by $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$.

The following lemma, can be deduced from assumption **(A)**.

Lemma 4.1. For every $p \geq 1$ and $t > 0$, there exists constant $C(p, t)$ such that for every $\varepsilon > 0$,

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} [|X_s^{1,\varepsilon}|^p + |X_s^{2,\varepsilon}|^p + |X_s^1|^p + |X_s^2|^p] \right) \leq C(p, t).$$

Proposition 4.2. Assume that **(A)**, **(B)** are satisfied. Let \bar{a} , \bar{b} , \bar{a}^n and \bar{b}^n be defined as in section 3. Let $X = (X^1, X^2)$ denote the solution of the SDE

$$X_s = x + \int_0^s \bar{b}(X_r) dr + \int_0^s \bar{\sigma}(X_r) dW_r, \quad 0 \leq s \leq t.$$

Then, for every $p \geq 1$,

$$(j) \quad \mathbb{E} \int_0^t |\bar{a}^n(X_r) - \bar{a}(X_r)|^p dr, \quad \longrightarrow 0 \text{ as } n \text{ tends to } \infty.$$

$$(jj) \quad \mathbb{E} \int_0^t |\bar{b}^n(X_r) - \bar{b}(X_r)|^p dr, \quad \longrightarrow 0 \text{ as } n \text{ tends to } \infty.$$

Proof. Proof of (j) and (jj). Let $N > 0$ and put $D_N := \{x \in \mathbb{R}^{d+1}, |x| \leq N\}$. For $(g, g^n) \in \{(\bar{a}, \bar{a}^n), (\bar{b}, \bar{b}^n)\}$, we have

$$\begin{aligned} \mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p dr &\leq 2^p \left(\mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p \mathbb{1}_{\{\sup_{s \leq r} |X_s| \leq N\}} dr \right. \\ &\quad \left. + \mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p \mathbb{1}_{\{\sup_{s \leq r} |X_s| > N\}} dr \right) \end{aligned}$$

Since \bar{g} and g^n satisfy **(A)**, **(B)**, there exists a constant C which is independent of n such that,

$$\begin{aligned} \mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p dr &\leq 2^p \left(\mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p \mathbb{1}_{\{\sup_{s \leq r} |X_s| \leq N\}} dr \right. \\ &\quad \left. + \frac{C}{N^p} \mathbb{E} \left(\sup_{0 \leq s \leq t} |X_s|^{2p} \right) \right) \end{aligned}$$

By Krylov's estimate, there exists a positive constant $K(t, N, d)$ which is independent of n such that

$$\mathbb{E} \int_0^t |g^n(X_r) - g(X_r)|^p dr \leq K(t, N, d + 1) \| |g^n - g|^p \|_{L^{d+1}(D_N)} + \frac{C}{N^p} \mathbb{E} \left(\sup_{0 \leq s \leq t} |X_s|^{2p} \right),$$

Passing successively to the limit in n and N , we get the desired result. ■

4.0.1 Tightness of the processes $(Y^\varepsilon, M^\varepsilon := \int Z_r^\varepsilon dM_r^{X^\varepsilon})$

Recall that the process Y^ε is defined by

$$Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^\varepsilon) dr - \int_s^t Z_r^\varepsilon dM_r^{X^\varepsilon}, \quad (4.1)$$

where $\bar{X}_s^{1,\varepsilon} = \frac{X_s^{1,\varepsilon}}{\varepsilon}$.

Proposition 4.3. There exists a positive constant C which does not depend on ε such that

$$\sup_\varepsilon \left\{ \mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon|^2 + \int_0^t |Z_s^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_s \right) \right\} \leq C. \quad (4.2)$$

Proof. Throughout this proof, K and C are positive constants which depend only on (s, t) and may change from line to line. According to Lemma 4.1 we have, for every $k \geq 1$,

$$\sup_{\varepsilon} \mathbb{E} \left(\sup_{0 \leq s \leq t} \left[|X_s^{1,\varepsilon}|^{2k} + |X_s^{2,\varepsilon}|^{2k} \right] \right) < +\infty. \quad (4.3)$$

Using Itô's formula, we get

$$\begin{aligned} |Y_s^\varepsilon|^2 + \int_s^t |Z_r^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_r &\leq |H(X_t^\varepsilon)|^2 + K \int_s^t |Y_r^\varepsilon|^2 dr + \int_s^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \\ &\quad + 2C \int_s^t |Y_r^\varepsilon| |Z_r^\varepsilon| dr - 2 \int_s^t \langle Y_r^\varepsilon, Z_r^\varepsilon dM_s^{X^\varepsilon} \rangle. \end{aligned}$$

Since $|\sigma(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon})|^2 = \text{Trace} \left(\sigma \sigma^* \left(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon} \right) \right) \geq c > 0$, one has

$$2C|Y_r^\varepsilon| |Z_r^\varepsilon| \leq C|Y_r^\varepsilon|^2 + \frac{1}{2}|Z_r^\varepsilon|^2 |\sigma(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon})|^2.$$

It follows that

$$\begin{aligned} \mathbb{E} \left(|Y_s^\varepsilon|^2 + \frac{1}{2} \int_s^t |Z_r^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_r \right) &\leq \mathbb{E} (|H(X_t^\varepsilon)|^2) + C_1 \mathbb{E} \left(\int_s^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \right) \\ &\quad + K \mathbb{E} \left(\int_s^t |Y_r^\varepsilon|^2 dr \right). \end{aligned}$$

According to Gronwall's Lemma, there exists a constant which does not depend on ε such that

$$\mathbb{E} (|Y_s^\varepsilon|^2) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \right), \quad \forall s \in [0, t].$$

We deduce that

$$\mathbb{E} \left(\int_s^t |Z_r^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_r \right) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \right) \quad (4.4)$$

Combining (4.4) and Burkholder-Davis-Gundy's inequality, we get

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon|^2 \right) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \right).$$

Hence,

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon|^2 + \frac{1}{2} \int_0^t |Z_r^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_r \right) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, 0, 0)|^2 dr \right)$$

In view of condition (C1-ii and iii) and inequality (4.3), the proof is complete. \blacksquare

Proposition 4.4. *Let $M_s^\varepsilon := \int_0^s Z_r^\varepsilon dM_r^{X^\varepsilon}$. The sequence $(Y^\varepsilon, M^\varepsilon)_{\varepsilon > 0}$ is tight on the space $\mathcal{D}([0, t], \mathbb{R}^L) \times \mathcal{D}([0, t], \mathbb{R}^L)$ endowed with the \mathbf{S} -topology.*

Proof. Since M^ε is a martingale, then according to [30] or [21], the Meyer-Zheng tightness criteria is fulfilled whenever

$$\sup_{\varepsilon} \left(CV(Y^\varepsilon) + \mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon| + |M_s^\varepsilon| \right) \right) < +\infty, \quad (4.5)$$

where CV denotes the conditional variation and is defined in appendix A.

Clearly

$$CV(Y^\varepsilon) \leq \mathbb{E} \left(\int_0^t |f(\bar{X}_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^\varepsilon)| ds \right).$$

Combining condition (C1) and Proposition 4.3, we derive (4.5). \blacksquare

4.0.2 A sequence of auxiliary processes, tightness and convergence.

For $n \in \mathbb{N}^*$, we define a sequence of an auxiliary process $Z^{\varepsilon, n}$ by

$$Z_s^{\varepsilon, n} := \nabla_x v^n(t-s, X_s^\varepsilon), \quad s \in [0, t] \quad (4.6)$$

We rewrite the process Y^ε in the form,

$$Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon, n}) dr + A_t^{\varepsilon, n} - A_s^{\varepsilon, n} - (M_t^\varepsilon - M_s^\varepsilon) \quad (4.7)$$

where

$$\begin{aligned} M_s^\varepsilon &:= \int_0^s Z_r^\varepsilon dM_r^{X^\varepsilon} \\ A_s^{\varepsilon, n} &:= \int_0^s [f(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon, n}) - f(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon, n})] dr. \end{aligned} \quad (4.8)$$

We define

$$\begin{aligned} \mathcal{M}_s^{\varepsilon, n} &:= \int_0^s Z_r^{\varepsilon, n} \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon}) dW_r \\ &\stackrel{(4.6)}{=} \int_0^s \nabla_x v^n(r, X_r^\varepsilon) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon}) dW_r \\ \mathcal{N}_s^{\varepsilon, n} &:= \int_0^s 1_{\{|Z_r^\varepsilon - Z_r^{\varepsilon, n}| > 0\}} \frac{(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})}{|(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})|} dW_r \\ \mathbf{L}_s^{\varepsilon, n} &:= \langle \mathcal{N}^{\varepsilon, n}, M^\varepsilon - \mathcal{M}^{\varepsilon, n} \rangle_s \\ &= \int_0^s 1_{\{|Z_r^\varepsilon - Z_r^{\varepsilon, n}| > 0\}} \frac{[(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})][(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})]^*}{|(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})|} dr \end{aligned}$$

Proposition 4.5. *For every $n \in \mathbb{N}^*$, the sequence $(\mathcal{M}^{\varepsilon, n}, \mathcal{N}^{\varepsilon, n}, A^{\varepsilon, n}, \mathbf{L}^{\varepsilon, n})_{\varepsilon > 0}$ is tight on the space $(\mathcal{C}([0, t], \mathbb{R}))^4$ endowed with the topology of uniform convergence.*

Proof. We prove the tightness of $(\mathbf{L}^{\varepsilon, n})_{\varepsilon > 0}$. Since $Z_s^{\varepsilon, n} := \nabla_x v^n(t-s, X_s^\varepsilon)$, then according to inequalities (4.2), (3.3) and (4.3), we have for any $n, p \in \mathbb{N}^*$:

$$\text{Max} \left(\sup_\varepsilon \mathbb{E} \int_0^t |Z_r^\varepsilon|^2 dr, \sup_\varepsilon \mathbb{E} \int_0^t |Z_r^{\varepsilon, n}|^2 dr, \sup_\varepsilon \mathbb{E} \sup_{0 \leq r \leq t} |X_r^{2, \varepsilon}|^p dr \right) < \infty. \quad (4.9)$$

We successively use assumption **(A2)** and Schwarz's inequality to show that for any n

$$\begin{aligned} \sup_\varepsilon \mathbb{E} \left(\sup_{|s'-s| \leq \delta} |\mathbf{L}_{s'}^{\varepsilon, n} - \mathbf{L}_s^{\varepsilon, n}| \right) &\leq \sup_\varepsilon \mathbb{E} \left(\sup_{|s'-s| \leq \delta} \int_s^{s'} |(Z_r^\varepsilon - Z_r^{\varepsilon, n}) \sigma(\bar{X}_r^{1, \varepsilon}, X_r^{2, \varepsilon})| dr \right) \\ &\leq K \sup_\varepsilon \mathbb{E} \left(\sup_{r \leq t} (1 + |X_r^{2, \varepsilon}|) \sup_{|s'-s| \leq \delta} \int_s^{s'} |(Z_r^\varepsilon - Z_r^{\varepsilon, n})| dr \right) \\ &\leq 2\sqrt{\delta} K \sup_\varepsilon \mathbb{E} \left(\sup_{r \leq t} (1 + |X_r^{2, \varepsilon}|) \left[\int_0^t (|Z_r^\varepsilon|^2 + |Z_r^{\varepsilon, n}|^2) dr \right]^{\frac{1}{2}} \right) \\ &\leq C\sqrt{\delta}. \end{aligned} \quad (4.11)$$

Using inequality (4.9) then letting δ tends to 0, we deduce the tightness of $(\mathbf{L}^{\varepsilon, n})_{\varepsilon > 0}$ from Theorem 7.3 in [6]. The tightness of $(A^{\varepsilon, n})_{\varepsilon > 0}$, $(\mathcal{M}^{\varepsilon, n})_{\varepsilon > 0}$ and $(\mathcal{N}^{\varepsilon, n})_{\varepsilon > 0}$ can be established by similar arguments. \blacksquare

Theorem 4.6. For every n , there exists a continuous process $(\mathcal{M}^n, \mathcal{N}^n, \mathbf{L}^n, A^n)$, a càd-làg process (\bar{Y}, \bar{M}) such that along a subsequence of ε , we have:

$(\mathcal{M}^{\varepsilon, n}, \mathcal{N}^{\varepsilon, n}, \mathbf{L}^{\varepsilon, n}, A^{\varepsilon, n}, Y^\varepsilon, M^\varepsilon) \Rightarrow (\mathcal{M}^n, \mathcal{N}^n, \mathbf{L}^n, A^n, \bar{Y}, \bar{M})$ on $(\mathcal{C}([0, t], \mathbb{R}))^4 \times (\mathcal{D}([0, t], \mathbb{R}))^2$ respectively endowed with the topology of the uniform convergence and the \mathbf{S} -topology.

Moreover there exists a countable subset \mathbf{D} of $[0, t]$ such that for any $k \geq 1, t_1, \dots, t_k \in \mathbf{D}$,

$$(Y_{t_1}^\varepsilon, M_{t_1}^\varepsilon, \dots, Y_{t_k}^\varepsilon, M_{t_k}^\varepsilon) \Rightarrow (\bar{Y}_{t_1}, \bar{M}_{t_1}, \dots, \bar{Y}_{t_k}, \bar{M}_{t_k}),$$

where \Rightarrow denotes the convergence in law.

Proof. From Propositions 4.4 and 4.5, the family $(\mathcal{M}^{\varepsilon, n}, \mathcal{N}^{\varepsilon, n}, \mathbf{L}^{\varepsilon, n}, A^{\varepsilon, n}, Y^\varepsilon, M^\varepsilon)_\varepsilon$ is tight on $(\mathcal{C}([0, t], \mathbb{R}))^4 \times (\mathcal{D}([0, t], \mathbb{R}))^2$, where the spaces are respectively endowed with the topology of the uniform convergence and the \mathbf{S} -topology. We deduce that along a subsequence (still denoted by ε), $(\mathcal{M}^{\varepsilon, n}, \mathcal{N}^{\varepsilon, n}, \mathbf{L}^{\varepsilon, n}, A^{\varepsilon, n}, Y^\varepsilon, M^\varepsilon)_\varepsilon$ converges in law on $(\mathcal{C}([0, t], \mathbb{R}))^4 \times (\mathcal{D}([0, t], \mathbb{R}))^2$ to a process $(\mathcal{M}^n, \mathcal{N}^n, \mathbf{L}^n, A^n, \bar{Y}^n, \bar{M}^n)$. The last statement follows from Theorem 3.1 in Jakubowski [21]. ■

4.0.3 The first identification of the limits in ε

In this subsection, we will determine the equation satisfied by the limit process (\bar{Y}, \bar{M}) .

Proposition 4.7. Let (\bar{Y}, \bar{M}) , be the process defined in Theorem 4.6 as a limit (as $\varepsilon \rightarrow 0$) of $(Y^\varepsilon, M^\varepsilon)$. Then,

(i) For every $s \in [0, t] - \mathbf{D}$,

$$\begin{cases} \bar{Y}_s = H(X_t) + \int_s^t \bar{f}(X_r^1, X_r^2, \bar{Y}_r, \nabla_x v^n(t-r, X_r)) dr + A_t^n - A_s^n - (\bar{M}_t - \bar{M}_s), \\ \mathbb{E}(\sup_{0 \leq s \leq t} |\bar{Y}_s|^2 + |X_s^1|^2 + |X_s^2|^2) \leq C. \end{cases} \quad (4.12)$$

(ii) Moreover, \bar{M} is \mathcal{F}_s^n -martingale, where $\mathcal{F}_s^n := \sigma\{X_r, \bar{Y}_r, \bar{M}_r, \mathcal{M}_r^n, \mathcal{N}_r^n, \mathbf{L}_r^n, A_r^n, 0 \leq u \leq s\}$ augmented with the \mathbb{P} -null sets.

To prove this proposition, we need some lemmas. The first one plays a similar role to that played by the invariant measure in the periodic case. It was introduced in [23] for a forward SDE and later adapted in [1] to systems of SDE-BSDE in which the generator of the backward component does not depend on the variable Z . We do not provide a proof, since that of Lemma 4.7 in [1] can be repeated word to word (also we have a new variable).

Lemma 4.8. Assume **(A)**, **(B)** and (C2)-(C4). For $(x_2, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d+1}$, let $V^\varepsilon(x, y, z)$ denote the solution of the PDE:

$$\begin{cases} a_{00}(\frac{x_1}{\varepsilon}, x_2) D_{x_1}^2 u(x, y, z) = f(\frac{x_1}{\varepsilon}, x_2, y, z) - \bar{f}(x, y, z), & x_1 \in \mathbb{R}, \\ u(0, x_2, y, z) = D_{x_1} u(0, x_2, y, z) = 0. \end{cases} \quad (4.13)$$

Then, for some bounded functions β_1 and β_2 satisfying (2.3) we have

(i) $D_{x_1} V^\varepsilon(x, y, z) = x_1(1 + |x_2|^2 + |y|^2 + |z|^2) \beta_1(\frac{x_1}{\varepsilon}, x_2, y, z)$,

and the same is true with $D_{x_1} V^\varepsilon$ replaced by $D_{x_1} D_{x_2} V^\varepsilon, D_{x_1} D_y V^\varepsilon$ and $D_{x_1} D_z V^\varepsilon$

(ii) $V^\varepsilon(x, y, z) = x_1^2(1 + |x_2|^2 + |y|^2 + |z|^2) \beta_2(\frac{x_1}{\varepsilon}, x_2, y, z)$,

and the same is true with V^ε replaced by $D_{x_2} V^\varepsilon, D_y V^\varepsilon, D_z V^\varepsilon, D_{x_2}^2 V^\varepsilon, D_y^2 V^\varepsilon, D_z^2 V^\varepsilon, D_{x_2} D_y V^\varepsilon, D_{x_2} D_z V^\varepsilon$ and $D_y D_z V^\varepsilon$.

Lemma 4.9. *We have, for any fixed $n \geq 1$,*

$$\sup_{0 \leq s \leq t} \left| \int_0^t \left(f\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, \nabla_x v^n(t-r, X_r^\varepsilon)\right) - \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, \nabla_x v^n(t-r, X_r^\varepsilon)) \right) dr \right|$$

tends to zero in probability as $\varepsilon \rightarrow 0$.

Proof. We set

$$h(\bar{X}_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n}) = f\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n}\right) - \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n}),$$

We shall show that for any $0 \leq s \leq t$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^s h(\bar{X}_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) dr \right| = 0$$

Let V^ε denote the solution of equation (4.13). Note that V^ε has first and second derivatives in (x, y, z) which are possibly discontinuous only at $x_1 = 0$. Then, as in [23], since φ^2 is bounded away from zero, we can use the Itô-Krylov formula to get

$$\begin{aligned} V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n}) &= V^\varepsilon(x, Y_0^\varepsilon, Z_0^{\varepsilon,n}) \\ &+ \int_0^s \left[f\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}\right) - \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) \right] dr \\ &+ \int_0^s \text{Trace} \left[\tilde{a}\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) D_{x_2}^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) \right] dr \\ &+ \int_0^s \left[D_{x_2} V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) \tilde{b}\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) - D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) f\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon\right) \right] dr \\ &+ \int_0^s \left[D_x V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) \sigma\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) + D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) Z_r^\varepsilon \sigma\left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) \right] dW_r \\ &+ \frac{1}{2} \int_0^s D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) Z_r^\varepsilon \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) (Z_r^\varepsilon)^* dr \\ &+ \frac{1}{2} \int_0^s D_x D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}\right) (Z_r^\varepsilon)^* dr \\ &+ \frac{1}{2} \int_0^s D_x D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle X^\varepsilon, Z^{\varepsilon,n} \rangle_r \\ &+ \frac{1}{2} \int_0^s D_y D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle Y^\varepsilon, Z^{\varepsilon,n} \rangle_r \\ &+ \frac{1}{2} \int_0^s D_z^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle Z^{\varepsilon,n} \rangle_r \\ &+ \int_0^s D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) dZ_r^{\varepsilon,n} \end{aligned} \tag{4.14}$$

In view of Lemma 4.8 and Proposition 4.3,

$$\lim_{\varepsilon \rightarrow 0} V^\varepsilon(x, Y_0^\varepsilon, Z_0^{\varepsilon,n}) = 0$$

Using the fact that $1 = 1_{\{|X_s^{1,\varepsilon}| < \sqrt{\varepsilon}\}} + 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}}$ and Lemma 4.8, we obtain

$$\begin{aligned} |V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n})| &\leq \varepsilon \left[(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2 + |Z_s^{\varepsilon,n}|^2) |\beta_2(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n})| \right] \\ &\quad + 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}} |X_s^{1,\varepsilon}|^2 [(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2 + |Z_s^{\varepsilon,n}|^2) \\ &\quad \beta_2(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n})| \end{aligned}$$

Thanks to Lemma 4.8 and Proposition 4.3, we deduce that

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n})| \right) \leq K \left(\varepsilon + \sup_{|x_1| \geq \sqrt{\varepsilon}} \sup_{(x_2, y, z)} |\beta_2(\frac{x^1}{\varepsilon}, x^2, y, z)| \right)$$

Since β_2 satisfies (2.3), the right hand side of the previous inequality tends to zero as $\varepsilon \rightarrow 0$. Similarly, one can show that each term on the lines from the third to the last one in the above identity tend to zero. Let us detail the arguments for the term on line six, and on the term on line eight. Let us start with the term on line 6, which is one of the most delicate ones.

$$\begin{aligned} &\left| \int_0^s D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) Z_r^\varepsilon \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) (Z_r^\varepsilon)^* dr \right| \\ &\leq C \sup_{0 \leq r \leq s} |D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n})| \text{Trace} \int_0^s Z_r^\varepsilon \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) (Z_r^\varepsilon)^* dr \end{aligned}$$

Since $\{\text{Trace} \int_0^s Z_r^\varepsilon \sigma \sigma^* (\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}) (Z_r^\varepsilon)^* dr, 0 \leq s \leq t\}$ is the increasing process associated to a martingale which is uniformly $L^1(\mathbb{P})$ -integrable, its square root has a bounded expectation. Moreover, arguing as for V^ε , one can show that

$$\sup_{0 \leq r \leq s} |D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n})| \text{ tends in probability to } 0 \text{ as } \varepsilon \rightarrow 0.$$

We now consider the term on line 8. Since $\nabla_x v^n(s, x) \in \mathcal{C}^{1,2}$, we use Itô's formula to get

$$\begin{aligned} \nabla_x v^n(0, X_t^\varepsilon) &= \nabla_x v^n(t, X_0^\varepsilon) + \int_0^t \Gamma(r, \frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}) dr \\ &\quad + \int_0^t D_{xx}^2 v^n(t-r, X_r^\varepsilon) \sigma \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) dW_r \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} \Gamma(r, \frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}) &= -\partial_r (\nabla_x v^n(t-r, X_r^\varepsilon)) - D_{xx}^2 v^n(t-r, X_r^\varepsilon) b \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) \\ &\quad + \frac{1}{2} \partial_{x,x,x}^3 v^n(t-r, X_r^\varepsilon) \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) \end{aligned}$$

According to inequalities (3.2) and (3.3), it follows that (4.15) is well-defined. Moreover, we have

$$\begin{aligned} &\frac{1}{2} \int_0^s D_x D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle X^\varepsilon, Z^{\varepsilon,n} \rangle_r \\ &\leq C \sup_{0 \leq r \leq s} |D_x D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n})| \times \int_0^s |\text{Trace} \sigma \sigma^* \left(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon} \right) D_{xx}^2 v^n(t-r, X_r^\varepsilon)| dr \end{aligned}$$

In view of condition **(A2)**, (4.3) and the fact that $|D_{xx}^2 v^n| \leq k_3^n$, the $L^p(\mathbb{P})$ norm of the increasing process $\int_0^s |\text{Trace } \sigma \sigma^* (\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}) D_{xx}^2 v^n(t-r, X_r^\varepsilon)| dr$ is bounded (by a constant not depending on ε), for each $p \geq 1$. Further, the same argument as above shows that

$$\sup_{0 \leq r \leq s} |D_x D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n})| \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0$$

Similarly, one can show that

$$\begin{aligned} & \frac{1}{2} \int_0^s D_y D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle Y^\varepsilon, Z^{\varepsilon,n} \rangle_r + \frac{1}{2} \int_0^s D_z^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) d\langle Z^{\varepsilon,n} \rangle_r \\ & + \int_0^s D_z V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) dZ_r^{\varepsilon,n} \end{aligned}$$

converges to zero in probability as ε tends to 0. The proof is complete. \blacksquare

Lemma 4.10. *For every $n \in \mathbb{N}^*$, the sequence of processes $\int_0^\cdot \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon, \nabla_x v^n(t-r, X_r^\varepsilon)) dr$ converges in law (as $\varepsilon \rightarrow 0$) to the process $\int_0^\cdot \bar{f}(X_r^1, X_r^2, \bar{Y}_r, \nabla_x v^n(t-r, X_r)) dr$ on $(\mathcal{C}([0, t], \mathbb{R}), \|\cdot\|_\infty)$.*

Proof. It can be performed as in [1]-Lemma 4.9. \blacksquare

Proof of Proposition 4.7 Passing to the limit in (4.7) and using Lemma 4.9 and Lemma 4.10, we derive assertion (i). Assertion (ii) can be proved by using the same argument as those of [34], section 6. \blacksquare

Let $\mathcal{F}_s^n := \sigma\{X_r, \bar{Y}_r, \bar{M}_r, \mathcal{M}_r^n, \mathcal{N}_r^n, \mathbf{L}_r^n, A_r^n, \quad 0 \leq u \leq s\}$ be the filtration generated by $(X, \bar{Y}, \bar{M}, \mathcal{M}^n, \mathcal{N}^n, \mathbf{L}^n, A^n)$ and completed by the \mathbb{P} -null sets. Combining the estimates in Proposition 4.3, inequality (4.3), Lemmas (A.3) and (A.4) in Appendix A, we show that \bar{M} is \mathcal{F}_s^n -martingale.

The following proposition summarizes Proposition 6.5.2 and Corollaries 6.5.3 and 6.5.4 in Delarue [12]. We will sketch the proof for the convenience of the reader.

Proposition 4.11. *For every $n \in \mathbb{N}^*$ and every $s \in [0, t]$ we have*

$$(i) \quad [\mathcal{N}^n, \bar{M} - \mathcal{M}^n]_s = \mathbf{L}_s^n.$$

(ii) *The process A^n is of bounded variation, and, for every progressively measurable process $\{\beta_s : 0 \leq s \leq t\}$ satisfying $\mathbb{E} \left(\int_0^t |\beta_r|^2 dr \right) < +\infty$ we have for any $0 \leq s \leq s' \leq t$,*

$$\left| \int_s^{s'} \langle \beta_r, dA_r^n \rangle \right|^2 \leq C \left(\int_s^{s'} |\beta_r|^2 dr \right) \left(\text{Trace} \left\{ [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s'} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_s \right\} \right) \quad (4.16)$$

Proof. We follow [12]. Assertion (i) is a consequence of Theorem 4.6. We prove assertion (ii). Thanks to (4.8) and assumption **C**, there exists $C > 0$ (which value may change from line to another) such that for every $\varepsilon > 0$, $n \in \mathbb{N}^*$ and $s \leq s' \leq t$:

$$|A_{s'}^{\varepsilon,n} - A_s^{\varepsilon,n}| \leq C \int_s^{s'} |Z_r^\varepsilon - Z_r^{\varepsilon,n}| ds$$

Using the definitions of M^ε , $\mathcal{M}^{\varepsilon,n}$, $\mathcal{N}^{\varepsilon,n}$ and the fact that the diffusion coefficient a is uniformly elliptic, we deduce that :

$$|A_{s'}^{\varepsilon,n} - A_s^{\varepsilon,n}| \leq C \text{trace}([\mathcal{N}^{\varepsilon,n}, M^\varepsilon - \mathcal{M}^{\varepsilon,n}]_{s'} - [\mathcal{N}^{\varepsilon,n}, M^\varepsilon - \mathcal{M}^{\varepsilon,n}]_s)$$

Using Theorem 4.6 and assertion (i), we show that for every $n \in N^*$ and $0 \leq s \leq s' \leq t$

$$|A_{s'}^n - A_s^n| \leq C \text{trace}([\mathcal{N}^n, \bar{M} - \mathcal{M}^n]_{s'} - [\mathcal{N}^n, \bar{M} - \mathcal{M}^n]_s)$$

Hence, thanks to the Kunita-Watanabe inequalities, for every progressively measurable process β , satisfying $\mathbb{E} \left(\int_0^t |\beta_r|^2 dr \right) < +\infty$

$$\left| \int_s^{s'} \langle \beta_r, dA_r^n \rangle \right| \leq C \left(\int_s^{s'} |\beta_r|^2 d \text{trace}[\mathcal{N}^n]_r \right)^{\frac{1}{2}} \left(\text{trace} \left\{ [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s'} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_s \right\} \right)^{\frac{1}{2}}$$

Since for every $\varepsilon > 0$ and $n \in N^*$, the process $(|\mathcal{N}^{\varepsilon, n}|^2 - s)$ is a supermartingale, then for every $n \in N^*$ the process $(|\mathcal{N}^n|^2 - s)$ is also a supermartingale. Following the proof of Theorem 4.10 of Chapter I in Kratzas & Shreve, we deduce that $(|\text{trace}([\mathcal{N}^n]_{s'}| - [\mathcal{N}^n]_s))$. This completes the proof of assertion (ii). \blacksquare

4.0.4 Identification of the limiting BSDE in n

For $s \in [0, t]$ we put

$$Y_s^n := v^n(t-s, X_s) \quad \text{and} \quad Z_s^n := \nabla_x v^n(t-s, X_s) \quad (4.17)$$

Proposition 4.12. *For every $s \in [0, t] - \mathbb{D}$,*

$$\lim_{n \rightarrow +\infty} \left(\mathbb{E} (|Y_s^n - \bar{Y}_s|) + \mathbb{E} \left\{ \left([\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_t - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_s \right) \right\} \right) = 0. \quad (4.18)$$

Proof. For $R > 0$, let $D_R := \{x \in \mathbb{R}^{d+1}, |x| \leq R\}$ and $\tau_R := \inf\{r > s, |X_r| > R\}$, $\inf\{\emptyset\} = \infty$.

Step 1: *Estimate of $\mathbb{E} (|Y_{s \wedge \tau_R}^n - \bar{Y}_{s \wedge \tau_R}|^2)$.*

By Itô's formula, we have

$$\begin{aligned} Y_s^n &= v^n(0, X_t) - \int_s^t \left[\frac{\partial v^n}{\partial r}(t-r, X_r) + \bar{L}v^n(t-r, X_r) \right] dr - \int_s^t \nabla_x v^n(t-r, X_r) dM_r^X \\ &= v^n(0, X_t) - \int_s^t \left[\frac{\partial v^n}{\partial r}(t-r, X_r) + \bar{L}^n v^n(t-r, X_r) \right] dr \\ &\quad + \int_s^t (\bar{L}^n - \bar{L}) v^n(t-r, X_r) dr - \int_s^t Z_r^n dM_r^X \end{aligned}$$

In view of (3.1), (4.12) and (4.17), we have

$$\begin{aligned} Y_s^n - \bar{Y}_s &= v^n(0, X_t) - \bar{Y}_t + \int_s^t [\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, \bar{Y}_r, Z_r^n)] dr \\ &\quad + \int_s^t (\bar{L}^n - \bar{L}) v^n(t-r, X_r) dr - \int_s^t dA_r^n + \int_s^t (d\bar{M}_r - Z_r^n dM_r^X) \end{aligned}$$

Using Itô's formula on $[s \wedge \tau_R, t \wedge \tau_R]$, it follows that

$$\begin{aligned} &\mathbb{E} (|Y_{s \wedge \tau_R}^n - \bar{Y}_{s \wedge \tau_R}|^2) + \mathbb{E} \left\{ \left([\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{t \wedge \tau_R} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s \wedge \tau_R} \right) \right\} \quad (4.19) \\ &= \mathbb{E} |v^n(0, X_{t \wedge \tau_R}) - \bar{Y}_{t \wedge \tau_R}|^2 + 2\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, \bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, \bar{Y}_r, Z_r^n) \rangle dr \\ &\quad + 2\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, (\bar{L}^n - \bar{L}) v^n(t-r, X_r) \rangle dr - 2\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, dA_r^n \rangle. \end{aligned}$$

On one hand, since \bar{f} is uniformly Lipschitz in the y -variable [thanks again to Assumption (C)-(i)], it follows (where the constant C can change from line to line),

$$\begin{aligned}
& 2\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, \bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, \bar{Y}_r, Z_r^n) \rangle dr \\
& \leq C\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |Y_r^n - \bar{Y}_r|^2 dr + \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 \\
& \leq C\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |Y_{r \wedge \tau_R}^n - \bar{Y}_{r \wedge \tau_R}|^2 dr + \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 \\
& \leq C\mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - \bar{Y}_{r \wedge \tau_R}|^2 dr + \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2
\end{aligned} \tag{4.20}$$

The same argument shows that

$$\begin{aligned}
& 2\mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, (\bar{L}^n - \bar{L}) v^n(t-r, X_r) \rangle dr \\
& \leq 2\mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - \bar{Y}_{r \wedge \tau_R}|^2 dr + \mathbb{E} \int_0^{t \wedge \tau_R} |\nabla_x v^n(t-r, X_r)|^2 |\bar{b}^n(X_r) - \bar{b}(X_r)|^2 dr \\
& \quad + \mathbb{E} \left(\int_0^{t \wedge \tau_R} |D_{xx}^2 v^n(t-r, X_r)|^2 |\bar{a}^n(X_r) - a(X_r)|^2 dr \right).
\end{aligned}$$

For each $n \in \mathbb{N}^*$ and $R > 0$, we put

$$\begin{aligned}
\delta_1^{n,R} & := \mathbb{E} |v^n(t-t \wedge \tau_R, X_{t \wedge \tau_R}) - \bar{Y}_{t \wedge \tau_R}|^2 + \mathbb{E} \int_s^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 dr \\
& \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |\nabla_x v^n(t-r, X_r)|^2 |\bar{b}^n(X_r) - \bar{b}(X_r)|^2 dr \\
& \quad + \mathbb{E} \left(\int_0^{t \wedge \tau_R} |D_{xx}^2 v^n(t-r, X_r)|^2 |\bar{a}^n(X_r) - a(X_r)|^2 dr \right).
\end{aligned}$$

In the other hand, we deduce from inequality (4.16), with the choice $\beta := Y^n - \bar{Y}$, that for any $\alpha > 0$,

$$\begin{aligned}
2\mathbb{E} \left| \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - \bar{Y}_r, dA_r^n \rangle \right| & \leq \frac{C}{\alpha^2} \mathbb{E} \left(\int_{s \wedge \tau_R}^{t \wedge \tau_R} |Y_r^n - \bar{Y}_r|^2 dr \right) \\
& \quad + C\alpha^2 \mathbb{E} \left(\left\{ [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{t \wedge \tau_R} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s \wedge \tau_R} \right\} \right). \\
& \leq \frac{C}{\alpha^2} \mathbb{E} \left(\int_s^t |Y_{r \wedge \tau_R}^n - \bar{Y}_{r \wedge \tau_R}|^2 dr \right) \\
& \quad + C\alpha^2 \mathbb{E} \left(\left\{ [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{t \wedge \tau_R} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s \wedge \tau_R} \right\} \right).
\end{aligned} \tag{4.21}$$

We choose α^2 such that $C\alpha^2 < \frac{1}{2}$ then we use identity (4.19) to get

$$\begin{aligned}
& \mathbb{E} (|Y_{s \wedge \tau_R}^n - \bar{Y}_{s \wedge \tau_R}|^2) + \frac{1}{2} \mathbb{E} \left\{ \left([\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{t \wedge \tau_R} - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_{s \wedge \tau_R} \right) \right\} \\
& \leq \delta_1^{n,R} + C\mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - \bar{Y}_{r \wedge \tau_R}|^2 dr.
\end{aligned}$$

Therefore, Gronwall's Lemma yields that

$$\begin{aligned} & \mathbb{E} (|Y_{s \wedge \tau_R}^n - \bar{Y}_{s \wedge \tau_R}|^2) + \mathbb{E} \left\{ \left([\bar{M} - \int_0^{\cdot} Z_r^n dM_r^X]_{t \wedge \tau_R} - [\bar{M} - \int_0^{\cdot} Z_r^n dM_r^X]_{s \wedge \tau_R} \right) \right\} \\ & \leq K_1(C, t) \delta_1^{n, R}. \end{aligned} \quad (4.22)$$

Step 2: $\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \delta_1^{n, R} = 0$.

We have $\delta_1^{n, R} = I_1^n + I_2^n + I_3^n$, with

$$\begin{aligned} I_1^n & := \mathbb{E} \int_0^{t \wedge \tau_R} |\nabla_x v^n(t-r, X_r)|^2 |\bar{b}^n(X_r) - \bar{b}(X_r)|^2 dr \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |D_{xx}^2 v^n(t-r, X_r)|^2 |\bar{a}^n(X_r) - a(X_r)|^2 dr, \end{aligned}$$

$$\begin{aligned} I_2^n & := \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 dr \\ & = \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, v^n(t-r, X_r), \nabla_x v^n(t-r, X_r)) - \bar{f}(X_r, v^n(t-r, X_r), \nabla_x v^n(t-r, X_r))|^2 dr, \end{aligned}$$

$$I_3^n := \mathbb{E} |v^n(t - t \wedge \tau_R, X_{t \wedge \tau_R}) - \bar{Y}_{t \wedge \tau_R}|^2.$$

Using Hölder's inequality, Krylov's estimate, (3.15) and Proposition 4.2, one can show that I_1^n tends to zero as n tends to infinity.

We show that I_2^n tends to 0 as n tends to ∞ . Let $M > 0$ and put $I_2^n := I_2^{n,1} + I_2^{n,2}$, with

$$I_2^{n,1} := \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 1_{\{|Y_r^n| + |Z_r^n| \leq M\}} dr$$

and

$$I_2^{n,2} := \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 1_{\{|Y_r^n| + |Z_r^n| > M\}} dr.$$

We have

$$I_2^{n,1} \leq \mathbb{E} \int_0^{t \wedge \tau_R} \sup_{\{|y| + |z| \leq M\}} |\bar{f}^n(X_r^1, X_r^2, y, z) - \bar{f}(X_r^1, X_r^2, y, z)|^2 dr.$$

We put $h^n(x) := \sup_{\{|y| + |z| \leq M\}} |\bar{f}^n(x, y, z) - \bar{f}(x, y, z)|$.

Thanks to Krylov's estimate, there exists a positive constant $N = N(t, R, d)$ such that

$$I_2^{n,1} \leq \mathbb{E} \int_0^{t \wedge \tau_R} |h^n(X_r)|^2 dr \leq N \|h^n\|_{L^{d+2}(D_R)}^2$$

Since \bar{f}^n and \bar{f} satisfy (C1), $Y_s^n := v^n(t-s, X_s)$ and $Z_s^n := \nabla_x v^n(t-s, X_s)$, we get

$$\begin{aligned}
I_2^{n,2} &\leq \mathbb{E} \int_0^{t \wedge \tau_R} (|\bar{f}^n(X_r, Y_r^n, Z_r^n)| + |\bar{f}(X_r, Y_r^n, Z_r^n)|)^2 1_{\{|Y_r^n| + |Z_r^n| > M\}} dr \\
&\leq 2K \mathbb{E} \int_0^{t \wedge \tau_R} (1 + |X_r| + |Y_r^n| + |Z_r^n|)^2 1_{\{|Y_r^n| + |Z_r^n| > M\}} dr \\
&\leq 2K \left(\mathbb{E} \int_0^{t \wedge \tau_R} (1 + |X_r| + |Y_r^n| + |Z_r^n|)^4 dr \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^{t \wedge \tau_R} 1_{\{|Y_r^n| + |Z_r^n| > M\}} dr \right)^{\frac{1}{2}} \\
&\leq \frac{2K}{M^{\frac{1}{2}}} \left(\mathbb{E} \int_0^{t \wedge \tau_R} (1 + |X_r|^4 + |Y_r^n|^4 + |Z_r^n|^4) dr \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^{t \wedge \tau_R} (|Y_r^n| + |Z_r^n|) dr \right)^{\frac{1}{2}} \\
&\leq \frac{2K}{M^{\frac{1}{2}}} \left(\mathbb{E} \int_0^{t \wedge \tau_R} (1 + |X_r|^4 + |v^n(t-r, X_r)|^4 + |\nabla_x v^n(t-r, X_r)|^4) dr \right)^{\frac{1}{2}} \\
&\quad \times \left(\mathbb{E} \int_0^{t \wedge \tau_R} (|v^n(t-r, X_r)| + |\nabla_x v^n(t-r, X_r)|) dr \right)^{\frac{1}{2}}
\end{aligned}$$

According to Krylov's estimate, there exists a constant $N = N(R, t, d)$ such that

$$\left(\mathbb{E} \int_0^{t \wedge \tau_R} (1 + |X_r|^4 + |v^n(t-r, X_r)|^4 + |\nabla_x v^n(t-r, X_r)|^4) dr \right)^{\frac{1}{2}} \leq N \left(1 + R + \|v^n\|_{L^{d+2}([0, t] \times D_R)}^4 + \|\nabla_x v^n\|_{L^{d+2}([0, t] \times D_R)}^4 \right)^{\frac{1}{2}}$$

and

$$\left(\mathbb{E} \int_0^{t \wedge \tau_R} (|v^n(t-r, X_r)| + |\nabla_x v^n(t-r, X_r)|) dr \right)^{\frac{1}{2}} \leq N \left(\|v^n\|_{L^{d+2}([0, t] \times D_R)} + \|\nabla_x v^n\|_{L^{d+2}([0, t] \times D_R)} \right)^{\frac{1}{2}}$$

But, thanks to (3.15), v^n and ∇v^n are bounded in each $L_{loc}^p([0, t] \times \mathbb{R}^{d+1})$ uniformly in n . We then deduce that there exists a positive constant $K_1 = K_1(t, R, d)$ such that

$$\sup_n I_2^{n,2} \leq \frac{K_1}{M^{\frac{1}{2}}}$$

Therefore,

$$I_2^n \leq K(t, R, d) \left[\|h^n\|_{L^{d+2}(D_R)}^2 + \frac{1}{M^{\frac{1}{2}}} \right] \quad (4.23)$$

Passing successively to the limit in n and M , we deduce that I_2^n tends to zero as n tends to infinity.

We shall show that I_3^n tends to 0 as n tends to ∞ . We have

$$\begin{aligned}
I_3^n &= \mathbb{E} |v^n(t-t \wedge \tau_R, X_{t \wedge \tau_R}) - \bar{Y}_{t \wedge \tau_R}|^2 \\
&= \mathbb{E} |v^n(t-t \wedge \tau_R, X_{t \wedge \tau_R}) - v(t-t \wedge \tau_R, X_{t \wedge \tau_R})|^2 + \mathbb{E} |v(t-t \wedge \tau_R, X_{t \wedge \tau_R}) - \bar{Y}_{t \wedge \tau_R}|^2
\end{aligned}$$

Since as R tends to ∞ , $v(t-t \wedge \tau_R, X_{t \wedge \tau_R})$ tends to $v(0, X_t) = H(X_t)$ and $\bar{Y}_{t \wedge \tau_R}$ tends to $\bar{Y}_t = H(X_t)$, then we pass to the limit first in n and next in R to deduce that I_3^n tends to zero as n tends to infinity. Consequently $\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \delta_1^{n,R} = 0$.

Since τ_R tends increasingly to infinity as R tends to infinity, then for R large enough $t \wedge \tau_R = t$ and hence $\lim_{n \rightarrow +\infty} \left(\mathbb{E} (|Y_s^n - \bar{Y}_s|) + \mathbb{E} \left\{ \left([\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_t - [\bar{M} - \int_0^\cdot Z_r^n dM_r^X]_s \right) \right\} \right) = 0$. \blacksquare

We now define

$$Y_s := v(t-s, X_s), \quad Z_s := \nabla_x v(t-s, X_s),$$

where v is the solution of the PDE (2.5). Note that although $\nabla_x v(\cdot, \cdot)$ is only an element of $L^p_{loc}([0, t] \times \mathbb{R}^{d+1})$ (for any $p \geq d+2$), since X is non degenerate diffusion, it follows from Krylov's estimate (see [24]) that $\nabla_x v(t-s, X_s)$ is well defined as a random element of $L^2(0, t)$.

Proposition 4.13. *For every $s \in [0, t]$,*

$$\lim_{n \rightarrow +\infty} \left(\mathbb{E}(|Y_s^n - Y_s|) + \mathbb{E} \int_s^t |Z_s^n - Z_s|^2 d\langle M^X \rangle_s \right) = 0$$

Proof. Since v belongs to $\mathcal{W}_{p,loc}^{1,2}$, then Itô–Krylov's formula and the uniqueness of the backward component of equation (2.4) show that for every $s \in [0, t]$,

$$Y_s = v(t-s, X_s) \tag{4.24}$$

In another hand, since

$$\begin{cases} Y_s = H(X_t) + \int_s^t \bar{f}(X_r, Y_r, Z_r) dr - \int_s^t Z_r dM_r^X \\ Y_s^n = v^n(0, X_t) - \int_0^t \bar{f}^n(X_r, v^n(t-r, X_r), \nabla_x v^n(t-r, X_r)) dr + \int_0^t (\bar{L}^n - \bar{L}) v^n(t-r, X_r) dr \\ \quad - \int_0^t Z_r^n dM_r^X \end{cases}$$

Using Itô's formula on $[s \wedge \tau_R, t \wedge \tau_R]$ then arguing as in the proof of Proposition 4.12, it holds that

$$\begin{aligned} & \mathbb{E}|Y_{s \wedge \tau_R}^n - Y_{s \wedge \tau_R}|^2 + \frac{1}{2} \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |Z_s^n - Z_s|^2 d\langle M^X \rangle_s \\ & \leq \mathbb{E} (|v^n(t - t \wedge \tau_R, X_{t \wedge \tau_R}) - Y_{t \wedge \tau_R}|^2) \\ & \quad + \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - Y_r, \bar{f}^n(X_r, v^n(t-r, X_r), v^n(t-r, X_r)) - \bar{f}(X_r, Y_r, Z_r) \rangle ds \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |(\bar{L}^n - \bar{L}) v^n(t-r, X_r)|^2 dr \\ & \quad + C \mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - Y_{r \wedge \tau_R}|^2 dr \end{aligned}$$

Since $(Y_t^n, Z_t^n) = (v^n(0, X_t), \nabla_x v^n(0, X_t))$, it follows that

$$\begin{aligned} & \mathbb{E}|Y_{s \wedge \tau_R}^n - Y_{s \wedge \tau_R}|^2 + \frac{1}{2} \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |Z_s^n - Z_s|^2 d\langle M^X \rangle_s \\ & \leq \mathbb{E} (|v^n(t - t \wedge \tau_R, X_{t \wedge \tau_R}) - Y_{t \wedge \tau_R}|^2) \\ & \quad + \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - Y_r, \bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n) \rangle ds \\ & \quad + \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - Y_r, \bar{f}(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r, Z_r^n) \rangle ds \\ & \quad + \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} \langle Y_r^n - Y_r, \bar{f}(X_r, Y_r, Z_r^n) - \bar{f}(X_r, Y_r, Z_r) \rangle ds \\ & \quad + \int_s^{t \wedge \tau_R} |(\bar{L}^n - \bar{L}) v^n(t-r, X_r)|^2 dr \\ & \quad + C \mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - Y_{r \wedge \tau_R}|^2 dr \end{aligned}$$

Since \bar{f} is uniformly Lipschitz in (y, z) with the same Lipschitz constants K as f , then for any $\alpha > 0$ satisfying $\frac{K}{\alpha} < \frac{1}{2}$, we have

$$\begin{aligned} & \mathbb{E}|Y_{s \wedge \tau_R}^n - Y_{s \wedge \tau_R}|^2 + \left(\frac{1}{2} - \frac{K}{\alpha}\right) \mathbb{E} \int_{s \wedge \tau_R}^{t \wedge \tau_R} |Z_r^n - Z_r|^2 d\langle M^X \rangle_r \\ & \leq \mathbb{E} (|v^n(t - t \wedge \tau_R, X_{t \wedge \tau_R}) - Y_{t \wedge \tau_R}|^2) \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 ds \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |(\bar{L}^n - \bar{L}) v^n(t - r, X_r)|^2 dr \\ & \quad + (C + K + \alpha) \mathbb{E} \int_s^t |Y_{r \wedge \tau_R}^n - Y_{r \wedge \tau_R}|^2 dr \end{aligned}$$

We set

$$\begin{aligned} \delta_2^{n,R} & := \mathbb{E} (|v^n(t - t \wedge \tau_R, X_{t \wedge \tau_R}) - Y_{t \wedge \tau_R}|^2) \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |\bar{f}^n(X_r, Y_r^n, Z_r^n) - \bar{f}(X_r, Y_r^n, Z_r^n)|^2 ds \\ & \quad + \mathbb{E} \int_0^{t \wedge \tau_R} |(\bar{L}^n - \bar{L}) v^n(t - r, X_r)|^2 dr \end{aligned}$$

Arguing as for $\delta_1^{n,R}$, we show that $\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \delta_2^{n,R} = 0$ and the conclusion follows as in the proof of Proposition 4.12. \blacksquare

Corollary 4.14. $\mathbb{P} \{\forall s \in [0, t], \bar{Y}_s = v(t - s, X_s)\} = 1$, which implies that $(\bar{Y}_s)_{s \leq t}$ is continuous. Moreover $Y^\varepsilon \Rightarrow Y$.

Proof. Combining Propositions 4.12 and 4.13, we deduce that for all $s \in [0, t] - D$, $\bar{Y}_s = Y_s = v(s, X_s)$ a.s. Hence \bar{Y} has a continuous modification, which coincides a.s. with Y on $[0, t]$. But \bar{Y} is càlàg, hence it is a.s. continuous and identical to Y .

Since \bar{Y} was defined as the limit in law of an arbitrary converging subsequence of the sequence Y^ε , $\bar{Y}_s = v(s, X_s)$, and the law of X is uniquely determined, the law of $\{v(s, X_s), 0 \leq s \leq t\}$ is uniquely determined. Consequently, the whole sequence converges : $Y^\varepsilon \Rightarrow Y$. \blacksquare

Proof of Corollary 2.5 From equations (4.7) and (4.12), we have

$$\begin{cases} Y_0^\varepsilon = H(X_t^\varepsilon) + A_t^{\varepsilon,n} + \int_0^t f(\bar{X}_r^\varepsilon, X_r^{2,\varepsilon}, Y_r^\varepsilon, Z_r^{\varepsilon,n}) dr - M_t^\varepsilon \\ \bar{Y}_0 = H(X_t) + A_t^n + \int_0^t \bar{f}(X_r, \bar{Y}_r, Z_r^n) dr - \bar{M}_t \end{cases}$$

By Corollary 4.14 and the continuity of the projection at the final time $t \notin D$: $y \mapsto y_t$, we deduce from the above two identities that Y_0^ε converges towards \bar{Y}_0 in distribution. Moreover, since $Y_0^\varepsilon, \bar{Y}_0$ are deterministic, we deduce that $\lim_{\varepsilon \rightarrow 0} Y_0^\varepsilon = \bar{Y}_0 = Y_0$. That is, by using the non simplified notation,

$$Y_0^{t,x,\varepsilon} \rightarrow Y_0^{t,x}.$$

In other words, as $\varepsilon \rightarrow 0$,

$$v^\varepsilon(t, x) \rightarrow v(t, x).$$

A Appendix: S-topology

The **S**-topology has been introduced by Jakubowski [21] as a topology defined on the Skorohod space of càdlàg functions: $\mathcal{D}([0, T]; \mathbb{R})$. This topology is weaker than the Skorohod topology but tightness criteria are easier to establish. These criteria are the same as the one used in Meyer-Zheng [30].

Let $N^{a,b}(z)$ denotes the number of up-crossing of the function $z \in \mathcal{D}([0, T]; \mathbb{R})$ in a given level $a < b$. We recall some facts about the **S**-topology.

Proposition A.1. *(A criteria for S-tight). A sequence $(Y^\varepsilon)_{\varepsilon>0}$ is S-tight if and only if it is relatively compact on the S-topology.*

Let $(Y^\varepsilon)_{\varepsilon>0}$ be a family of stochastic processes in $\mathcal{D}([0, T]; \mathbb{R})$. Then this family is tight for the S-topology if and only if $(\|Y^\varepsilon\|_\infty)_{\varepsilon>0}$ and $(N^{a,b}(Y^\varepsilon))_{\varepsilon>0}$ are tight for each $a < b$.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis. If $(Y)_{0 \leq t \leq T}$ is a process in $\mathcal{D}([0, T]; \mathbb{R})$ such that Y_t is integrable for any t , the conditional variation of Y is defined by

$$CV(Y) = \sup_{0 \leq t_1 < \dots < t_n = T, \text{partition of } [0, T]} \sum_{i=1}^{n-1} \mathbb{E}[|\mathbb{E}[Y_{t_{i+1}} - Y_{t_i} | \mathcal{F}_{t_i}]|].$$

The process is call *quasimartingale* if $CV(Y) < +\infty$. When Y is a \mathcal{F}_t -martingale, $CV(Y) = 0$. A variation of Doob inequality (cf. lemma 3, p.359 in Meyer and Zheng [30], where it is assumed that $Y_T = 0$) implies that

$$\begin{aligned} \mathbb{P} \left[\sup_{t \in [0, T]} |Y_t| \geq k \right] &\leq \frac{2}{k} \left(CV(Y) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t| \right] \right), \\ \mathbb{E} \left[N^{a,b}(Y) \right] &\leq \frac{1}{b-a} \left(|a| + CV(Y) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t| \right] \right). \end{aligned}$$

It follows that a sequence $(Y^\varepsilon)_{\varepsilon>0}$ is S-tight if

$$\sup_{\varepsilon>0} \left(CV(Y^\varepsilon) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^\varepsilon| \right] \right) < +\infty.$$

Theorem A.2. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a S-tight family of stochastic process in $\mathcal{D}([0, T]; \mathbb{R})$. Then there exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ decreasing to zero, some process $Y \in \mathcal{D}([0, T]; \mathbb{R})$ and a countable subset $D \in [0, T]$ such that for any n and any $(t_1, \dots, t_n) \in [0, T] \setminus D$,*

$$(Y_{t_1}^{\varepsilon_k}, \dots, Y_{t_n}^{\varepsilon_k}) \xrightarrow{Dist} (Y_{t_1}, \dots, Y_{t_n})$$

Remark A.1. The projection $:\pi_T y \in (\mathcal{D}([0, T]; \mathbb{R}), S) \mapsto y(T)$ is continuous (see Remark 2.4, p.8 in Jakubowski,1997), but $y \mapsto y(t)$ is not continuous for each $0 \leq t \leq T$.

Lemma A.3. *Let $(Y^\varepsilon, M^\varepsilon)$ be a multidimensional process in $\mathcal{D}([0, T]; \mathbb{R}^p)$ ($p \in \mathbb{N}^*$) converging to (Y, M) in the S-topology. Let $(\mathcal{F}_t^{X^\varepsilon})_{t \geq 0}$ (resp. $(\mathcal{F}_t^X)_{t \geq 0}$) be the minimal complete admissible filtration for X^ε (resp. X). We assume that $\sup_{\varepsilon>0} \mathbb{E} [\sup_{0 \leq t \leq T} |M_t^\varepsilon|^2] < C_T \forall T > 0$, M^ε is a $\mathcal{F}^{X^\varepsilon}$ -martingale and M is a \mathcal{F}^X -adapted. Then M is a \mathcal{F}^X -martingale.*

Lemma A.4. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a sequence of process converging weakly in $\mathcal{D}([0, T]; \mathbb{R}^p)$ to Y . We assume that $\sup_{\varepsilon>0} \mathbb{E} [\sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2] < +\infty$. Hence, for any $t \geq 0$, $E [\sup_{0 \leq t \leq T} |Y_t|^2] < +\infty$.*

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