Optimization of light structures: the vanishing mass conjecture

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Abstract: We consider the shape optimization problem which consists in placing a given mass m of elastic material in a design region so that the compliance is minimal. Having in mind optimal light structures, our purpose is to show that the problem of finding the stiffest shape configuration simplifies as the total mass m tends to zero: we propose an explicit relaxed formulation where the compliance appears after rescaling as a convex functional of the relative density of mass. This allows us to write necessary and sufficient optimality conditions for light structures following the Monge-Kantorovich approach developed recently in [5].

1. Introduction

Since the beginning of the mathematical theory of elasticity it was possible to consider from a rigorous point of view the problem of finding the structure that, for a given system f of loads, gives the best resistance in terms of minimal compliance. In other words, an elastic structure is optimal if the corresponding displacement u is such that the total work $\int f \cdot u \, dx$ is minimal. However, even if the setting of the problem does not require particular mathematical tools, only in the last two decades there has been a deep understanding of *shape optimization problems* from a mathematical point of view. This was mainly due to the dramatic improvement in the field impressed by the powerful theories of homogenization and Γ-convergence which have been developed meanwhile.

What became clear soon was that in a large number of situations the optimal shape does not exist, and the existence of an optimal solution must be intended only in a *relaxed sense*. The form of the relaxed optimization problem was first studied (see [22,23]) in the so called *scalar case* where the physical problem only involves state variables with value in $\mathbb{R}$, like the problem of optimal mixtures of two given conductors. In this case the relaxed solutions have been completely studied, and identified as symmetric matrices with bounded and measurable coefficients, whose eigenvalues satisfy some suitable *bounds*. A similar result was also obtained in the elasticity problem (see for instance [16]) for optimal mixtures of two homogeneous and isotropic materials. In almost all cases which have been considered, the optimal relaxed solution is not isotropic and this was interpreted by saying that an optimal shape does not exist and minimizing sequences are composed by *laminates*.

We want to emphasize that the case of optimal elastic structures, or also simply the study of optimal shapes of a given conductor, seems to have an additional difficulty with respect to the problem of optimal mixtures. Indeed, the first correspond to the case of optimal mixtures when one of the two materials (or conductors) has the elasticity constants (or the conductivity coefficient) equal to zero. In this case, due to a lack of uniform ellipticity, it is known that among all possible relaxed problems, obtained as limits of sequences of elliptic problems on classical domains, there are some that are not of *local* type, and it is not clear if these *nonlocal* relaxed solutions could be optimal. This interesting direction of research has been developed recently in the scalar case in [2] and [20] showing deep connections with the theory of Dirichlet forms, and more recently in the case of elasticity in [14].

Here we are interested in a apparently more difficult problem which consists in finding the asymptotic of the previous shape optimization problem when the total volume tends to zero. In other words we are trying to give a mathematical fundation for what we call optimal light structures. In [1] it has been proposed to solve this problem by following three steps: 1) describe optimal mixtures of two elastic materials (homogenization) 2) Pass to the limit when the rigidity constants of the weak material tend to zero, 3) Pass to the limit as the ratio of void goes to 1 (high porosity limit).

However I got the impression that this approach is very heavy and, as far as I know, in view of the difficulties explained above no real mathematical justification has been given until now for what concerns steps 2 and 3. In addition the occurrence of concentration on lower dimensional structures expected in many cases by engineers and manufacturers pushes a priori for searching optimality out of the class of microstructures.

The aim of this paper is to propose a new direct approach leading to a very simple formula for the limit compliance where the light structure is described in term of the density distribution of material. This density is a possibly concentrated non negative measure and the corresponding energy functional turns out to be *convex*. As a consequence we may use the framework I recently developed in collaboration with G.Buttazzo and P. Seppecher [4, 5, 10] which allows us to see the optimal measure as the multiplier of a linear programming problem and also to write necessary and sufficient optimality conditions (Monge-Kantorovich system).

The plan of the paper is the following. In Section 2 we present the rescaled shape optimization problem associated with a small total mass ε. It is written as the minimization of a functional $c_\varepsilon(\mu)$ defined on probability measures. Then we state the form of the Γ-limit of the sequence $\{c_\varepsilon\}$ as $\varepsilon \to 0$. The model of Michell truss like structures is recovered as a particular case in dimension 2. In section 3, we introduce a framework suitable for dealing with lower dimensional structures in $\mathbb{R}^n$. We derive a relaxed compliance for structures whose dimension is prescribed to be less than or equal to $k < n$. Applying this result for $k = n-1$ allows us to prove the upperbound inequality for the Γ-limit of $\{c_\varepsilon\}$. In the last section, the lower bound inequality is presented as a consequence of what we call the *vanishing mass conjecture*. Some geometrical arguments are given to support this conjecture.

2. Setting of the problem and the vanishing mass model.

Notations. In what follows Ω is a bounded Lipschitz connected open subset of $\mathbb{R}^n$, Σ is a compact subset of $\overline{\Omega}$ and F is an element of $\mathcal{M}(\overline{\Omega};\mathbb{R}^n)$, the class of all $\mathbb{R}^n$-valued measures on $\overline{\Omega}$ with finite total variation.

The class of smooth displacements we consider is the Schwartz space $\mathcal{D} := \mathcal{D}(\mathbb{R}^n;\mathbb{R}^n)$ of C^∞ functions with compact support; similarly, the notation $\mathcal{D}'(\mathbb{R}^n;\mathbb{R}^n)$ stands for the space of vector valued distributions and, for a given nonnegative measure μ, $L^2_\mu(\mathbb{R}^n;\mathbb{R}^d)$ denotes the space of p-integrable functions. The symbol $\cdot$ stands for the Euclidean scalar product between two vectors in $\mathbb{R}^n$ or between two matrices in $\mathbb{R}^{n^2}$.

The elastic structure is placed in Ω and occupies a region (an open subset) $\omega \subset \Omega$ of prescribed volume m. It is clamped on the part of ω in contact with Σ and it has to support the given load F. The (possibly infinite) compliance associated with this configuration is given by

$$c(\omega, F, \Sigma) := -\inf\left\{\int_\omega j(e(u))dx- < F, u > :\ u \in \mathcal{D},\ u = 0 \text{ on } \Sigma\right\} , \tag{2.1}$$

where $j : \mathbb{R}^{n^2}_{\rm sym} \mapsto [0,+\infty)$ is a quadratic form characterizing the elastic properties of the material and $e(u)$ denotes the symmetrized tensor of deformations i.e. $e(u)_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. In the 3D case a model example is given by the isotropic elasticity with Lamé coefficients α, β $(3\alpha + 2\beta > 0)$:

$$j(z) = \frac{1}{2}\alpha|trz|^2 + \beta|z|^2 . \tag{2.2}$$

A classical situation is when Σ is a part of the boundary of Ω and F is of the kind

$$< F, u >= \int_\Omega fudx + \int_{\partial\Omega\setminus\Sigma} gu\, d\mathcal{H}^{n-1} \qquad \text{being } f \in L^2(\Omega;\mathbb{R}^3)\ ,\ g \in L^2(\partial\Omega\setminus\Sigma;\mathbb{R}^3).$$

In this case and if the data f, g are compatible (their supports need to be contained in $\overline{\omega}$), the infimum in (2.1) is finite and minimizers can be searched in the Sobolev space $W^{1,2}(\omega;\mathbb{R}^n)$. Let us stress the fact that much more general situations will be considered in our framework since we intend further to consider concentrated loads (for example Dirac delta) as well as sets ω whose measure becomes very small.

Our interest is to pass to the limit as $m \to 0$ in the following variational problem

$$\inf\{c(\omega)\ ;\ |\omega| = m\} . \tag{2.3}$$

As the data Σ, F are kept fixed all along the paper we will write $c(\omega, F, \Sigma) = c(\omega)$.

Rescaled Problem. We set the total mass m to be a small parameter ε tending to 0. It is easy to check that the infimum in (2.3) scales as $\frac{1}{\varepsilon}$ as $\varepsilon \to 0$. Indeed by making the change of variables $v = \varepsilon u$ in the integral, we obtain the identity:

$$\varepsilon\, c(\omega) = c_\varepsilon(\mu) \quad , \qquad \mu(x) = \frac{1}{\varepsilon}\, 1_\omega(x) ,$$

where 1_ω denotes the characteristic function of ω and c_ε is the functional on $\mathcal{M}_+(\overline{\Omega})$ defined by:

$$c_\varepsilon(\mu) := \begin{cases} -\inf\left\{\int j(e(u)d\mu - <F,u>: \ u \in \mathcal{D}, u = 0 \text{ on } \Sigma\right\} \\ \text{if } \mu = \mu(x)\,dx \ , \mu(x) \in \{0, \frac{1}{\varepsilon}\} \quad (+\infty \ \text{ otherwise}) \ . \end{cases} \tag{2.4}$$

The total mass constraint becomes $\int_{\overline{\Omega}} d\mu = 1$ so that (2.3) can be restated for $m = \varepsilon$ and after rescaling as the minimization of $c_\varepsilon(\mu)$ over all probability measures on μ on $\overline{\Omega}$. It is clear that the latter infimum decreases if we relax the constraint on the density of μ which appears in (2.4). More precisely, we have

$$\inf\{c_\varepsilon(\mu) \ : \ \int_{\overline{\Omega}} d\mu = 1\} \ \geq \ \inf\{c(\mu) \ : \ \int_{\overline{\Omega}} d\mu = 1\} \ , \tag{2.5}$$

where $c(\mu)$ is defined for every $\mu \in \mathcal{M}_+(\overline{\Omega})$ by

$$c(\mu) := -\inf\left\{\int j(e(u)d\mu - <F,u>: \ u \in \mathcal{D}, u = 0 \text{ on } \Sigma\right\} \ . \tag{2.6}$$

We notice that the functional $c(\mu)$ defined above has the form already used in [5] to modelize mass optimization problems. It is a convex lower semicontinous functional on measures. Unfortunately, as we will see later, a gap in (2.5) will occur in general except particular situations where the original mechanical problem can be reduced to a scalar setting.

Now the natural procedure in order to pass to the limit as $\varepsilon \to 0$ in the left hand side of (2.5) consists in computing the Γ-limit of c_ε with respect to the weak (star) convergence of measures on the compact $\overline{\Omega}$. Let us introduce the following integrand:

$$\bar{j}(z) := \sup\left\{z \cdot \xi - j^*(\xi) \ : \ \xi \in \mathbb{R}^{n^2}_{\text{sym}} \ , \ det\,\xi = 0\right\} \ , \tag{2.7}$$

where j^* denotes the Fenchel transform of ξ^* (i.e. $j^*(\xi) = \sup\{z \cdot \xi - j(z) \ : \ z \in \mathbb{R}^{n^2}_{\text{sym}}\}$). By Lemma 3.1 in section 3, it turns out that $\bar{j}$ is convex continuous and satisfies

$$\bar{j}(z) \leq j(z) \quad \text{for all } z \quad , \quad (\bar{j})^*(\xi) = j^*(\xi) \quad \text{whenever } rank(\xi) < n \ .$$

Our claim is two fold: first we say that the Γ limit of the sequence $\{c_\varepsilon\}$ exists and is a convex functional; then we claim that this limit can be represented like in (2.6) but substituting j with the new integrand $\bar{j}$. Precisely we conjecture the following result:

Theorem 2.1. *(Conjecture) The Γ-limit of c_ε with respect to the weak convergence of measures on $\overline{\Omega}$ is the functional $E(\mu)$ defined on $\mathcal{M}_+(\overline{\Omega})$ by*

$$E(\mu) := -\inf\left\{\int \bar{j}(e(u)d\mu - <F,u>: \ u \in \mathcal{D}, u = 0 \ \textit{on } \Sigma\right\} \ . \tag{2.8}$$

To prove this theorem, we need to show that:

a) For every sequence $\{\mu_\varepsilon\}$ such that $\mu_\varepsilon \rightharpoonup \mu$, there holds $\liminf_\varepsilon c_\varepsilon(\mu_\varepsilon) \geq E(\mu)$.

b) For every measure $\mu \in \mathcal{M}_+(\overline{\Omega})$, there exists a sequence $\{\mu_\varepsilon\}$ such that $\mu_\varepsilon \rightharpoonup \mu$ and $\limsup_\varepsilon c_\varepsilon(\mu_\varepsilon) \leq E(\mu)$.

The proof of b) will be sketched in section 3 where structures of lower dimension are considered. The proof of a) will be straightforward in the case where μ is concentrated on lower dimensional manifolds. The general case will be seen as a consequence of geometrical properties related with sets of vanishing measure (in section 4, we call it *vanishing mass conjecture*).

Let us now compute $\bar{j}$ for particular functions j given in the form (2.2).

a) Assume $\alpha = 0, \beta = \frac{1}{2}$ and $n = 3$. Then, we can exploit formula (2.7) writing the symmetric tensor ξ in the form $\xi = \tau_1 e \otimes e + \tau_2 e^\perp \otimes e^\perp$, where e is a unit vector and $\tau_1, \tau_2, 0$ are the eigenvalues of ξ. We obtain

$$\begin{aligned}\bar{j}(z) &= \sup\left\{\tau_1(ze\cdot e) + \tau_2(ze^\perp\cdot e^\perp) - \frac{1}{2}(\tau_1^2+\tau_2^2) \ :\ |e|=1,\ \tau_1,\tau_2\in\mathbb{R}\right\}\\ &= \sup\left\{\frac{1}{2}\left((ze\cdot e)^2 + (ze^\perp\cdot e^\perp)^2\right) \ :\ |e|=1\right\}\\ &= \frac{1}{2}(\lambda_1(z)^2+\lambda_2(z)^2)\ ,\end{aligned}$$

where $|\lambda_1(z)| \geq |\lambda_2(z)| \geq |\lambda_3(z)|$ are the eigenvalues of the tensor z.

The computation of $(\bar{j})^*$ is rather complicated and of course can be generalized for any pair of Lamé coefficients α, β. In fact we recover this way the formulae obtained in [1] where explicit form of the relaxed stress potential are given (these formulae turn out to be in agreement with our $(\bar{j})^*$).

b) The case $n = 2$ is simpler. Setting $\gamma = \frac{\alpha+2\beta}{4\beta(\alpha+\beta)}$, we have that $j^*(\xi) = \frac{1}{2}\gamma\tau^2$ holds for any rank one tensor of the kind $\xi = \tau e\otimes e$ where $|e| = 1$. Then denoting by $|\lambda_1(z)| \geq |\lambda_2(z)|$ the eigenvalues of $z \in \mathbb{R}^4_{\rm sym}$ and by $\tau_1(\xi), \tau_2(\xi)$ the eigenvalues of ξ we find easily

$$\bar{j}(z) = \frac{1}{2\gamma}\ |\lambda_1(z)|^2 \quad , \quad (\bar{j})^*(\xi) = \frac{1}{2}\gamma\ (|\tau_1(\xi)| + |\tau_2(\xi)|)^2. \tag{2.9}$$

We notice that in both examples the new potential $\bar{j}$ is not quadratic any more. However it remains always convex and homogeneous of degree 2. Accordingly we are able to treat the minimization of the compliance relative to $\bar{j}$ using many tools developed in [5]. This is summarized in the following corollary. It is convenient to introduce the following convex continuous positively 1-homogeneous integrands on $\mathbb{R}^{n^2}_{\rm sym}$:

$$\rho(z) := \inf\{t>0 : \bar{j}(\frac{z}{t}) \leq \frac{1}{2}\} \quad , \quad \rho^0(\xi) := \sup\{\xi\cdot z \ :\ \bar{j}(z)\leq \frac{1}{2}\}\ .$$

As $\bar{j}$ is 2-homogeneous, we have $\bar{j}(z) = \frac{1}{2}\,\rho(z)^2$ and $\bar{j}^*(\xi) = \frac{1}{2}\,(\rho^0(\xi))^2$ An important quantity associated with the data Ω, F, Σ is given by:

$$I(F,\Omega,\Sigma) := \sup\{< F, u > :\ \rho(e(u)) \leq 1\ ,\ u \in U^\infty(\Omega;\mathbb{R}^n),\ \ u = 0 \text{ on}\Sigma\}\ , \tag{2.10}$$

where $U^\infty(\Omega;\mathbb{R}^n)$ denotes the space of functions $u \in L^\infty(\Omega;\mathbb{R}^n)$ such that $e(u) \in L^\infty(\mathbb{R}^n;\mathbb{R}^{n^2}_{\rm sym})$. In [5] (see also [6]) it is proved that the latter supremum is achieved. Notice that in general functions in U^∞ are not Lipschitz due to the lack of Korn's inequality in $W^{1,\infty}$.

Corollary 2.2. *Let $\{\omega_\varepsilon\}$ a sequence of domains such that*

$$c(\omega_\varepsilon) \le \inf\{c(\omega) : |\omega| = \varepsilon\} + O(\varepsilon) \qquad \textit{(c.f. (2.3))}\ .$$

Then the sequence $\mu_\varepsilon = \frac{1_{\omega_\varepsilon} dx}{\varepsilon}$ converges weakly (up to a subsequence) to a probability measure μ on $\overline{\Omega}$. The following assertions hold:

i) μ solves the minimum problem:

$$\inf\left\{E(\mu)\ :\ \operatorname{spt}\mu \subset \overline{\Omega},\ \int d\mu = 1\right\} \qquad (\ F \textit{ given by (2.8)})\ . \tag{2.11}$$

ii) We have:

$$\min \textit{(2.11)} = \frac{\big(I(F,\Omega,\Sigma)\big)^2}{2}\ .$$

ii) The following equality holds:

$$I(F,\Omega,\Sigma) = \min\left\{\int \rho^0(\lambda)\ :\ \lambda \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^{n^2}_{\rm sym})\ ,\ -\operatorname{div}\lambda = F \textit{ on } \mathbb{R}^n \setminus \Sigma\right\}\ . \tag{2.12}$$

Moreover if a vector measure λ is optimal for (2.12), then $\mu = I(F,\Omega,\Sigma)\ \rho^0(\lambda)$ is a minimizer of problem (2.11).

Remark 2.3. The measure λ appearing in (2.12) is the stress field associated with the equilibrium of the optimal structure. This stress can be written in the form $\lambda = \sigma\mu$ where the density σ satisfies $\rho^0(\sigma)$ is constant along the optimal structure represented by the measure μ. This fact is the mathematical counterpart of some fact which is well known to engineers. Recall here that the notation $\rho^0(\lambda)$ denotes the non negative measure of density $\rho^0(\sigma)$ with repect to μ. By the 1-homogeneity of ρ^0 this measure is independent of the decomposition $\lambda = \sigma\mu$ (see [17]). In the case $n = 2$, owing to (2.9) we obtain $\rho^0(\xi) = \sqrt{\gamma}(|\tau_1(\xi)| + |\tau_2(\xi)|)$ and the infimum problem (2.12) becomes nothing else but the celebrated Michell's problem [21].

Remark 2.4. The optimal solution μ of (2.11) can be characterized by a system of optimality conditions (Monge-Kantorovich system, see [5]) which involves a notion of μ-tangential derivative. It turns out that in general the solution μ is not unique and is very sensitive to the form of the integrand $\bar{j}$ or equivalently to the form of the convex set of matrices $\{z \in \mathbb{R}^{n^2}_{\rm sym} : \rho(z) \le 1\}$. In fact using the optimality conditions, it has been proved in [5, example 5.1] that for some 2D configurations, the minimum (2.11) can be strictly greater than the one obtained keeping the initial elastic potential $j(z) = |z|^2$ instead of $\bar{j}(z) = \lambda_1(z)^2$. Moreover the topology of the solution changes drastically: the numerical solution for j found in [18] gives a two dimensional positive density (no holes) whereas for $\bar{j}$ (Michell's problem) a lot of solutions made with junction of bars can be found.

Remark 2.5. The same problem can be handled from the beginning in the scalar case (heat equation), meaning that in (2.1), $u : \mathbb{R}^n \to \mathbb{R}$, $j : \mathbb{R}^n \to \mathbb{R}$ and $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R})$. In this case, the situation becomes much simpler: Theorem 2.1 holds with $E(\mu) \equiv c(\mu)$. In other words $\overline{j} = j$ and the gap appearing in (2.5) goes to zero as $\varepsilon \to 0$. It turns out that in this case as $\varepsilon \to 0$, minimizing structures either concentrate on lower dimensional manifolds or spread in many thin layers which are parallel to the direction of the gradient of some optimal u related to problem (2.10). Let us finally notice that in the case $\Sigma = \emptyset$ and $\int F = 0$, the supremum in (2.10) is nothing else but the Monge -Kantorovich norm distance between the positive and negative parts of F.

3. Compliance of lower dimensional structures.

These lower dimensional structure can be justified from two sides:

- they can be seen as limits of n-dimensional structures supported on sets of vanishing measure (this is the essence of the classical fattening approach);
- the designer can be interested in finding stiff structures made exclusively with beams (1D structures) or with plates (2D structures) or also with a blend of 1D-2D structures (excluding volumic parts). In other words, the competitors μ in the minimization problem with respect to $c(\mu)$ given in (2.11) have to be searched in the subclass of measures supported by subsets of dimension k with $k = 1$ or $k = 2$ or $k \leq 2$. Of course this additional constraint will increase the value of the infimum.

We introduce for every value of the integer k ($k \in [1, n]$) the following functional on $\mathcal{M}(\overline{\Omega})$:

$$c_k(\mu) \ := \begin{cases} c(\mu) & \text{if } \dim T_\mu(x) = k \ \mu\text{-a.e.} \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

where $\dim T_\mu(x)$ represents the dimensional of the tangent space to μ at x. This notion will be made precise later.

We introduce also the following convex integrand on $\mathbb{R}^{n^2}_{\text{sym}}$:

$$j_k(z) := \sup \left\{ z \cdot z' - j^*(\xi) \ : \ \xi \in \mathbb{R}^{n^2}_{\text{sym}} \ , \ \operatorname{rank} \xi \leq k \right\} \ . \tag{3.2}$$

The properties of the j_k's are summarized in the following lemma

Lemma 3.1.

i) For every k, j_k is convex, homogeneous of degree two and we have

$$0 = j_0 \leq j_1 \leq \ldots \leq j_{n-1} \leq j_n = j \quad \textit{and} \quad j_{n-1} = \overline{j} \ .$$

ii) The Fenchel conjugate of j_k is given by

$$j_k^*(\xi) = \inf \left\{ \int j^* d\nu \ : \operatorname{spt} \nu \subset G_k, \ [\nu] = \xi \right\} \ , \tag{3.3}$$

where the infimum is taken on probability measures on $\mathbb{R}^{n^2}_{\rm sym}$*,* $[\nu]$ *is the barycenter of* ν *and* G_k *denotes the tensors of rank not greater that* k*.*

iii) We have $j_k^*(\xi) = j^*(\xi)$ *whenever* $\xi \in G_k$*.*

Proof. i) is trivial and iii) is a consequence of ii) noticing that $\nu = \delta_\xi$ is admissible when ξ belongs to G_k. Let us denote now by $g_k(\xi)$ the infimum which appears in the right hand side of (3.3). Clearly g_k is convex and since any symmetric tensor $\xi \in \mathbb{R}^{n^2}_{\rm sym}$ can be decomposed in a convex combination of at most n rank one tensors, it is finite and continuous. Therefore proving the equality $j_k^* = g_k$ is equivalent to showing that $g_k^* = j_k$. For every $z \in \mathbb{R}^{n^2}_{\rm sym}$, we have

$$
\begin{aligned}
g_k^*(z) &= \sup_{\xi,\nu} \left\{ z\cdot\xi - \int j^* d\nu \ : \ [\nu] = \xi,\ \operatorname{spt}\nu \subset G_k \right\} \\
&= \sup_{\nu} \left\{ \int \big(z\cdot s - j^*(s)\big) d\nu \ : \operatorname{spt}\nu \subset G_k \right\} \ \le j_k(z)\ .
\end{aligned},
$$

where the last inequality becomes an equality if we choose competitors ν of the kind $\nu = \delta_\xi$ where ξ runs over G_k. ■

The main result of this section is the following:

Theorem 3.2. *Let* $\bar c_k$ *denote the lower semicontinous envelope of* c_k*. Then there holds for every measure* $\mu \in \mathcal{M}_+(\overline{\Omega})$*:*

$$
\bar c_k(\mu) = -\inf \left\{ \int j_k(e(u) d\mu - < F, u >: \ u \in \mathcal{D}, u = 0 \ on \ \Sigma \right\} \ . \tag{3.4}
$$

Remark 3.3. We stress that the domain of $\overline{c_k}$ extends to all the space $\mathcal{M}_+(\overline{\Omega})$. This means that the dimensional constraint is not closed; however the approximation of higher dimension structures (say of dimension l) involves some additional energy which corresponds to the gap between the integrands j_k and j_l (see lemma 3.1). Roughly speaking a structure obtained by using micro-structures made of beams might look like a 3D elastic structure. However its compliance cannot be predicted by using the original model of 3D elasticity.

Remark 3.4. The relaxation procedure for c_k does not change if we replace the equality constraint $\dim T_\mu(x) = k$ μ-a.e. by the inequality constraint constraint $\dim T_\mu(x) \le k$. The lower semicontinuous envelope of this new functional c_k will be still given by the right hand side of (3.4). On the other hand, the identity $\bar j = j_{n-1}$ (see assertion i) of Lemma 3.1) implies that we have, for every μ:

$$
E(\mu) \ = \ \overline{c_{n-1}}(\mu)\ . \tag{3.5}
$$

Corollary 3.5. *For every probability measure μ on $\overline{\Omega}$, there exists a sequence of subsets $A_\varepsilon \subset \Omega$ such that $\mu_\varepsilon \rightharpoonup \mu$ and* $\limsup c_\varepsilon(\mu_\varepsilon) \leq E(\mu)$.

Proof. (sketch) Take μ of the kind $\mu = \theta \mathcal{H}^{n-1} \llcorner S$ where S is a smooth $n-1$-dimensional manifold and θ is a continuous and positive weight on S. Then using a standard fattening method, it is possible to construct a sequence of sets A_ε such that $|A_\varepsilon| = \varepsilon$, $\mu_\varepsilon := \frac{1_{A_\varepsilon}}{\varepsilon} \rightharpoonup \mu$ and such that $\limsup_\varepsilon c_\varepsilon(\mu_\varepsilon) = \limsup_\varepsilon c(\mu_\varepsilon) \leq c(\mu)$. Therefore the Γ-limsup of c_ε (denote it E_+) which is weakly lower semicontinous satisfies for every such $n-1$ dimensional measure μ the inequality

$$E_+(\mu) \leq c(\mu) = c_{n-1}(\mu) \ .$$

The previous inequality is then extended to all $n-1$ dimensional measures yielding the inequality $E_+ \leq c_{n-1}$. The conclusion follows by passing to the lower semicontinuous envelopes taking into account (3.5) . ∎

The end of this section is devoted to a sketch of the proof of Theorem 3.2 (a complete version will be found soon in [3]). Before we need to give a precise meaning to the notion of k dimensional structure and for that we make use of the concept of tangent space $T_\mu(x)$ to a measure introduced in [10] which makes sense for *any* positive Borel measure μ on $\mathbb{R}^n$. The underlying idea is to identify every subset S of $\mathbb{R}^n$ having Hausdorff dimension k with the overlying measure $\mathcal{H}^k$, possibly weighted by a positive density θ; more in general, a multijunction made by the union of sets S_i with different dimensions k_i may be described through a positive measure μ of the kind $\sum_i \theta_i \mathcal{H}^{k_i} \llcorner S_i$. We refer to the papers [4,7,8, 9,10,11,12,13] where this framework has been developed with many applications in elasticity, shape optimization and homogenization.
Here we only sketch the features which are useful for the understanding of some arguments developed later.

Tangent space . It is a μ- measurable multifunction $T_\mu(x)$ from $\mathbb{R}^n$ into the linear subspaces of $\mathbb{R}^n$. The shortest way to define it (perhaps not the more intuitive one) is the following: consider the operator $B : (L^2_\mu)^n \mapsto L^2_\mu$ defined by

$$\begin{cases} D(B) := \left\{ \sigma \in (L^2_\mu)^n \ : \ \exists C > 0 \text{ such that } \left| \int \sigma \cdot \nabla u \, d\mu \right| \leq C \|u\|_{2,\mu} \ \forall u \in \mathcal{D} \right\} \\ B\sigma = v \iff -\operatorname{div}(\sigma\mu) = v\mu \ . \end{cases}$$

Then it can be proved that the closure of $D(B)$ coincide with the set of selections of a unique (up to the μ a.e. equivalence class) of a multifunction. This multifunction denoted $T_\mu(x)$ (our tangent space) is characterized by the following equality

$$\overline{D}(B) = \left\{ \sigma \in L^2_\mu(\mathbb{R}^n; \mathbb{R}^n) \ : \ \sigma(x) \in T_\mu(x) \ \mu \text{a.e.} \right\} \ . \tag{3.6}$$

In what follows $P_\mu(x)$ (possibly identified to an element of $\mathbb{R}^{n^2}_{\text{sym}}$) will denote the orthogonal projector on $T_\mu(x)$.

Tangential gradient . A more intuitive path to reach the definition of $T_\mu(x)$ is motivated by the following problem: given a sequence $\{u_h\} \subset \mathcal{D}$ such that $(u_h, \nabla u_h) \to (u, \chi)$ in L^2_μ, what can we say of the relation between u and χ ? In other words, we are looking for a characterization of the closure of the set

$$G := \{u, \nabla u) \ : \ u \in \mathcal{D}(\mathbb{R}^n)\} \ .$$

It turns out that we have

$$(3.7) \qquad (u, \chi) \in \overline{G} \iff u \in H^1_\mu \quad \text{and} \quad \exists \xi \in L^2_\mu(\mathbb{R}^n; T^\perp_\mu) \ : \chi = \nabla_\mu u + \xi \ .$$

Here $\nabla_\mu u$ (μ-*tangential gradient*) is defined for smooth u by setting $\nabla_\mu u(x) = P_\mu(\nabla u(x))$ and is extended in a unique way (as a closable operator) to all functions of the Sobolev space H^1_μ. This space H^1_μ is the completion of smooth function with respect to the Hilbert norm $\|u\| := (\int (u^2 + |\nabla_\mu u|^2) d\mu)^{\frac{1}{2}}$

Tangential strain . The same scheme can be used substituting ∇u by the strain $e(u)$ of a vector function $u \in \mathcal{D}(\mathbb{R}^n; \mathbb{R}^n)$ (see [5,13]. We now consider:

$$G := \left\{(u, e(u)) \in (L^2_\mu)^n \times L^2_\mu(\mathbb{R}^n; \mathbb{R}^{n^2}_{\text{sym}}) \ : \ u \in \mathcal{D}(\mathbb{R}^n; \mathbb{R}^n)\right\} \ .$$

The analogous of statement (3.7) reads as

$$(3.8) \qquad (u, \chi) \in \overline{G} \iff u \in \mathcal{D}^{1,2}_\mu \quad \text{and} \quad \exists \xi \in L^2_\mu(\mathbb{R}^n; M^\perp_\mu) \ : \chi = e_\mu(u) + \xi \ .$$

Here $M_\mu(x)$ is a multifunction from $\mathbb{R}^n$ to vector subspaces of $\mathbb{R}^{n^2}_{\text{sym}}$ which can be defined using the analogous of operator B defined in (3.6) where now $\sigma \in L^2_\mu(\mathbb{R}^n; \mathbb{R}^{n^2}_{\text{sym}})$. In [12], it is shown that under very mild regularity assumptions on μ (regularity by blow-up), the relation between M_μ and T_μ is explicit:

$$(3.9) \qquad M_\mu(x) = \left\{P_\mu(x)\xi P_\mu(x) \ : \ \xi \in \mathbb{R}^{n^2}_{\text{sym}}\right\} \ .$$

Denoting by $Q_\mu(x)$ the orthogonal projector on $M_\mu(x)$, the tangential strain is defined for C^1 functions by $e_\mu(x) = [Q_\mu(x)](e(u)(x))$ extended by continuity to $\mathcal{D}^{1,2}_\mu$ the completion of the space of smooth deformations with respect to the L^2_μ-energy.

Stress formulation for the compliance It is a matter of classical convex analysis to show that $c(\mu)$ given by (2.11) can also be written as

$$(3.10) \quad c(\mu) = \min\left\{\int j^*(\sigma)\, d\mu \ : \ \sigma \in L^2_\mu(\mathbb{R}^n; \mathbb{R}^{n^2}_{\text{sym}}) \ , \ -\operatorname{div}(\sigma\mu) = F \quad \text{on } \mathbb{R}^n \setminus \Sigma\right\} \ .$$

We stress the fact that the competitors σ in (3.10) belong to $D(B)$ and therefore satisfies $\sigma(x) \in M_\mu(x)$, μ a .e. In particular if μ is associated with a k-dimensional structure, then we have rank $\sigma(x) \leq k$ μ a .e

We notice also that a similar representation formula for $\overline{c_k}(\mu)$,$E(\mu)$ are obtained by simply substituting j^* respectively with j^*_k and $(\bar{j})^*$ in (3.10).

Proof of Theorem 3.2 (sketch):

i) *Lowerbound:* Denote by $E_k(\mu)$ the right hand side of (3.4). It is a convex l.s.c. functional and we have

$$E_k(\mu) = \min \left\{ \int j_k^*(\sigma)\, d\mu \ : \ \sigma \in L^2_\mu(\mathbb{R}^n; \mathbb{R}^{n^2}_{\text{sym}}) \ , \ -\operatorname{div}(\sigma\mu) = F \quad \text{on } \mathbb{R}^n \setminus \Sigma \right\} \ .$$

Assume that $\dim(T_\mu(x)) \leq k$. Then arguing as above, the competitors σ have rank $\leq k$ and therefore $j_k^*(\sigma) = j^*(\sigma)$. Owing to (3.10), we deduce that, for such measures μ, there holds $E_k(\mu) = c(\mu) = c_k(\mu)$. Thus the inequality $c_k \geq E_k$ holds over all $\mathcal{M}_+(\overline{\Omega})$. Passing to the lower semicontinuous envelopes, we deduce that $\overline{c_k} \geq E_k$.

ii) *Upperbound:* We restrict here to the case where μ is the Lebesgue measure. For any Y-periodic measure ν of dimension $\leq k$ (here Y is the possibly rotated unit cube in $\mathbb{R}^n$ and we normalize ν so that $\nu(Y) = 1$), we consider $\mu_\varepsilon := \nu(\frac{x}{\varepsilon})$ which clearly converges to μ. The limit of $c(\mu_\varepsilon)$ can be predicted from the theory of homogenization for thin structures which has been developed in several directions (see [15] and also [8,12,11]). The upperbound inequality for $\overline{c_k}$ can be deduced by optimizing other such periodic measures ν using in particular the homogenized stress potential representation obtained in [12] together with an argument of localization which allows to commute infimum and integral. ∎

4. Lowerbound inequality and the vanishing mass conjecture.

To complete the proof of Theorem 2.1, we need to show that for every sequence of sets $\{A_\varepsilon\}$ such that $|A_\varepsilon| = \varepsilon$, the following implication holds:

$$\mu_\varepsilon := \frac{1_{A_\varepsilon}}{\varepsilon} \rightharpoonup \mu \quad \Rightarrow \quad \liminf_\varepsilon c_\varepsilon(\mu_\varepsilon) \geq E(\mu) \ . \tag{4.1}$$

Note that in general $c(\mu) \leq E(\mu)$ (see (3.4) whereas the lower semicontinuity of $c(\mu)$ implies only the inequality $\liminf_\varepsilon c_\varepsilon(\mu_\varepsilon) = \liminf_\varepsilon c(\mu_\varepsilon) \geq c(\mu)$.

In fact the inequality (4.1) is straightforward in the case where μ has no n-dimensional part. More precisely, we have

Lemma 4.1. *Let $\mu \in \mathcal{M}_+(\overline{\Omega})$ such that $\dim T_\mu(x) < n$, μ a.e. Then*

$$\Gamma - \lim_\varepsilon c_\varepsilon(\mu) = E(\mu) \ .$$

Proof. According to the observation made above, we are done if we show that such measures satisfy $E(\mu) = c(\mu)$. We argue on the stress formulations of $c(\mu)$ and $E(\mu)$ (see (3.10), simply noticing that all admissible stress fields σ in the minimum problem (3.10) satisfy $\sigma \in M_\mu(x)$ and therefore $\operatorname{rank} \sigma(x) \leq \dim(T_\mu(x)) \leq n-1$ μ a.e. Thus by Lemma 3.1, we have $(\bar{j})^*(\sigma) = j^*_{n-1}(\sigma) = j^*(\sigma)$ yielding the equality $E(\mu) = c(\mu)$. ∎

The approach we suggest for attacking the general case consists, like in previous lemma, in considering the dual formulations. Let σ_ε be the optimal stress related to μ_ε ; it satisfies

$$c_\varepsilon(\mu_\varepsilon) = \int j^*(\sigma_\varepsilon) d\mu_\varepsilon \quad , \quad -\operatorname{div} \sigma_\varepsilon\, \mu_\varepsilon = F \quad \text{on } \mathbb{R}^n \setminus \Sigma \ .$$

Assuming that $c_\varepsilon(\mu_\varepsilon)$ is uniformly bounded (otherwise (4.1) is trivial), by the growth condition on j we have that

$$\sup_\varepsilon \int |\sigma_\varepsilon|^2 \, d\mu_\varepsilon \ < \ +\infty \ . \tag{4.2}$$

From (4.2) it follows that possibly passing to subsequences, there exists a suitable $\sigma \in L^2_\mu(\mathbb{R}^n; \mathbb{R}^{n^2}_{\rm sym})$ such that $\sigma_\varepsilon\, \mu_\varepsilon \rightharpoonup \sigma\, \mu$ and $-\operatorname{div}\, \sigma\, \mu = f$ on $\mathbb{R}^n \setminus \Sigma$. Then showing (2.1) reduces to establish that

$$\liminf_\varepsilon \int j^*(\sigma_\varepsilon)\, d\mu_\varepsilon \ \geq \ \int (\bar{j})^*(\sigma)\, d\mu \ . \tag{4.3}$$

Clearly we need more information on the oscillatory behaviour of matrix fields σ_ε and we therefore consider a family of Young measures associated with the triple $\{\sigma_\varepsilon, \mu_\varepsilon, \mu\}$. It can be shown (see [9, Prop 4.3]) that there exists a suitable family of probability measures ν_x on $\mathbb{R}^{n^2}_{\rm sym}$ (defined μ a.e.x) such that:

$$\Psi(\sigma_\varepsilon)\, \mu_\varepsilon \ \rightharpoonup \ \left(\int \Psi(\xi) \nu_x(d\xi)\right) \mu \quad \text{for all } \Psi \in C_0(\mathbb{R}^{n^2}_{\rm sym}) \quad . \tag{4.4}$$

Testing (4.4) in the particular case $\Psi(\xi) = \xi$ gives the equality $\sigma = [\nu_x]$ between the weak limit σ introduced above and the barycenter of ν_x. Now our task is to provide arguments for establishing the following inequality for every element ν of the Young family $\{\nu_x\}$ generated by $\{\sigma_\varepsilon\}$ (we omit further the index x) :

$$\int j^*(\xi)\, d\nu \geq (\bar{j})^*([\nu]) \ . \tag{4.5}$$

Together with (4.4), (4.5) implies the lower bound inequality (4.3).

A natural guess for proving (4.5) is to conjecture that ν is supported on the subset of tensors with rank $< n$, yielding that $j^*(\xi) = (\bar{j})^*(\xi)$ holds ν a.e. This property suggests that if the sets A_ε oscillate at a small scale (so that for example $\frac{1_{A_\varepsilon}}{\varepsilon}$ converges weakly to the Lebesgue measure), then they need to concentrate at the same scale on lower dimensional manifolds. The reason for this is that A_ε becomes thinner and thinner whereas, due to the divergence condition, the unit exterior normal $n_\varepsilon(x)$ satisfies $\sigma_\varepsilon n_\varepsilon(x) = 0$ on $\partial A_\varepsilon \setminus sptF$.

Unfortunately this picture is not the good one as we can see on the following example in $\mathbb{R}^2$ suggested by P. Seppecher [24]: Take $\Omega = Y := (-1/2, 1/2)^2$ and r_ε chosen so that $\pi r_\varepsilon^2 = \varepsilon$. Denote by χ_ε the Y-periodization of the characteristic function of the ball $\{|y| < r_\varepsilon\}$. Then $\chi(\frac{x}{\varepsilon})$ determines a subset A_ε of Ω such that $|A_\varepsilon| \sim \varepsilon$ and $\mu_\varepsilon = \frac{1_{A_\varepsilon}}{\varepsilon} dx \rightharpoonup \mu$

being μ the Lebesgue measure on Ω. Now it is easy to construct a Y- periodic function $\varphi_\varepsilon \in L^\infty(Y; \mathbb{R}^4_{sym})$ such that div $\varphi_\varepsilon = 0$, $\varphi_\varepsilon(y) = 0$ if $|y| \geq 2\, r_\varepsilon$ and $\varphi_\varepsilon(y) = I_2$ if $|y| \leq r_\varepsilon$ here I_2 denotes the identity matrix). Then the sequence $\sigma_\varepsilon = \varphi_\varepsilon(\frac{x}{\varepsilon})$ is divergence free and satisfies (4.2). However the Young family $\{\nu_x\}$ defined through (4.4) turns out to be independent on x and has a Dirac mass concentrated on the matrix I_2.

However we notice that in the previous example the weak limit of $\sigma_\varepsilon \mu_\varepsilon$ is zero so that (4.5) still holds. In fact the property (4.5) is far from requiring that the support of ν contains only degenerate tensors. For example, (4.5) is satisfied if ν can be decomposed as a convex combination of probability measures $\nu = t\nu_0 + (1-t)\nu_1$ where $\mathrm{spt}(\nu_0) \subset \{\det \xi = 0\}$ and $det([\nu_1]) = 0$. This can be easily checked recalling that j^* and $\bar{j}^*$ agree on determinant free tensors and by making use of Jensen's inequality.

Basically what we call "*vanishing mass conjecture*" consists in saying that the property (4.5) is satisfied *for every convex function j on* $\mathbb{R}^{n^2}_{\mathrm{sym}}$ and for all Young measures generated by sequences $\{\sigma_\varepsilon\}$ considered above. In particular we claim that the validity of (4.1) or (4.5) is not related to the fact that j is quadratic (although it could be helpful to use tools like compensated compactness or H-measures). To conclude let us give an intrinsic way to express (4.5) independently of j which has been suggested by P.Seppecher [24]:

Conjecture: *The probability ν can be decomposed as follows: there exists a probability measure ν_0 on $\{\det \xi = 0\}$ and for ν_0 almost all ξ, there exists a probability measure λ_ξ on $\mathbb{R}^{n^2}_{\mathrm{sym}}$ such that $[\lambda_\xi] = \xi$ and for all $\Psi \in C_0(\mathbb{R}^{n^2}_{\mathrm{sym}})$:*

$$\int \Psi d\nu = \int \left(\int \Psi(y) \lambda_\xi(dy) \right) \nu_0(d\xi) \ . \tag{4.6}$$

As before it is easy to check that (4.6) implies (4.5). Indeed by using two times Jensen's inequality and the fact that $j^* = (\bar{j})^*$ ν_0 a.e., we obtain:

$$\begin{aligned} \int j^*(\xi)\, d\nu &\geq \int \left(\int j^*(y) \lambda_\xi(dy) \right) \nu_0(d\xi) \geq \int j^*(\xi)\, \nu_0(d\xi) \\ &\geq \int (\bar{j})^*(\xi)\, \nu_0(d\xi) \geq (\bar{j})^*([\nu]) \ . \end{aligned}$$

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