HOMOGENIZATION OF GRAPH-BASED ELASTIC STRUCTURES

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Abstract

In the framework of Γ-convergence and periodic homogenization of highly contrasted materials, we study cylindrical structures made of one material and voids. Interest in high contrast homogenization is growing rapidly but assumptions are generally made in order to remain in the framework of classical elasticity. On the contrary, we obtain homogenized energies taking into account second gradient (i.e. strain gradient) effects. We first show that we can reduce the study of the considered structures to discrete systems corresponding to frame lattices. Our study of such lattices differs from the literature in the fact that we must take into account the different orders of magnitude of the extensional and flexural stiffnesses. This allows us to consider structures which would have been floppy when considering only extensional stiffness and completely rigid when considering flexural stiffnesses of the same order of magnitude than the extensional ones. At our knowledge, this paper provides the first rigorous homogenization result with a complete second gradient limit energy.

1 Introduction

In [14] it has been proved that highly contrasted heterogeneous elastic materials may lead, through an homogenization process, to materials with very new properties. In particular the order of differentiation of the equilibrium equations may be much higher for the homogenized material than they were for the heterogeneous one. However very few explicit examples have been given in which such a phenomenon appears. In [31], [9], [12] the homogenized material becomes a second gradient one : the elastic energy depends on the second gradient of the displacement instead of the first one only. However all these results fall under the framework of couple stress theory, [36], [37], [27], [28] : the dependence with respect to the second gradient of the displacement is limited to dependence on the gradient of the skew part of the gradient of the displacement only. To our knowledge complete second gradient media have been obtained, up to now, only through homogenization of discrete systems based on pantographic structures [5], [35], [5].

Second gradient materials are, among other generalized continuum models widely used [18], [17], [25], [19]. Their very rich behavior allows for instance to regularize and thus to study precisely the parts of materials where the deformation tends to concentrate, [38], [20] (inter-phases, [29], [13], [21], [34], porous media, [33], fractures, [2], [3], damage and plasticity, [39], [32], . . .). However the second gradient properties are scarcely measured directly, [7], [8], [22] nor rigorously interpreted from a microscopic point of view. Mechanicians have no tool for conceiving second gradient materials with chosen properties.

The aim of this paper is to provide such a tool. It is is not question here to solve all highly contrasted periodic homogenization problems but to describe a set of situations sufficiently large for making clear how appear second gradient effect through the homogenization process.

We consider structures made of a periodic arrangement of welded thin walls (see for instance figure 1) : they are cylinders the basis of which is a thickened periodic planar graph. We study, in the framework of Γ-convergence, the homogenization of these structures and rigorously determine the second gradient effects. To that aim we make some modeling assumptions which, of course, can be questioned when applied to the real structure of figure 1:

- First we assume that the structure is made of a homogeneous linear elastic material. We thus implicitly forbid the possibility of any micro buckling effect.

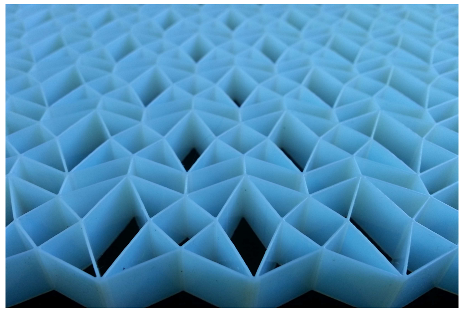

Figure 1: A cylindrical 3D elastic structure based on a thickened periodic planar graph.

- We also consider that the structure is solicited in the plane of the graph and assume that we are in the conditions of plane strain elasticity. This assumption, valid only when the height of the structure is large enough, allows us to reduce the problem to a bi-dimensional one : a linear elastic problem set in a thickened periodic planar graph; more precisely, in the intersection of this thickened graph and a bounded domain $\Omega \subset \mathbb{R}^2$.

- As our goal is to determine the effective properties of the material, we have to suppose that the size ℓ of the period of the graph is small compared to the characteristic size L of the domain Ω. This is the standard asymptotic homogenization assumption.
$$\epsilon = \ell/L << 1$$

- We consider that the thickness e of the walls the structure is made of (i.e. the thickness of the graph) is small compared with ℓ. Hence the 2D elastic problem we consider contains two small dimensionless parameters which we let tend to zero :
$$\delta = e/\ell << 1$$
This assumption is essential : otherwise the standard homogenization results would be valid and the effective properties of the material would be those of a classical (may be non isotropic) elastic material. This assumption will also have a practical effect on our mathematical arguments : as it implies that the edges are slender rectangles, we can, using the theory of slender elastic structures, reduce our problem to the study of a discrete system.

- The two limits $\epsilon \to 0$ and $\delta \to 0$ do not commute and we have to specify the way they simultaneously go to zero : we assume that
$$\delta = \beta\epsilon$$
with $\beta > 0$ fixed. Indeed, this case is critical : the cases $\delta = \epsilon^\alpha$ with $\alpha > 1$ or $\alpha < 1$ can be deduced from our results by letting in a further step β tend to zero or to infinity.

- Finally we have to specify the order of magnitude of the rigidity of the material our structure is made of. We emphasize that speaking of the order of magnitude of the stiffness of the material takes sense only if we compare it to some force. In other words, making an assumption over the elastic rigidity is equivalent to making an assumption over the order of magnitude of the applied external forces. As the total volume of our structure tends to zero with δ, it is clear that we need a strong rigidity of the material if we desire to resist to forces of order one. Different assumptions can be made which correspond to different experiments. This is not surprising : the reader accustomed for instance to the 3D-2D or 3D-1D reduction of models for plates or beams, knows that changing the assumptions upon the order of magnitude of the elasticity stiffness of the material changes drastically the limit model. If the structure cannot resist to some applied forces (like a membrane cannot resist to transverse forces), it may resist to them after a suitable scaling of the material properties (like the membrane model is replaced by the Kirchhoff-Love plate model). Simultaneously some mobility may disappear (like the Kirchhoff-Love plate becomes inextensible).

  In this paper we are interested in the case where the Lamé coefficients (μ, λ) of the material tend to infinity like $\delta^{-1}\epsilon^{-2}$:
$$\mu = \frac{\mu_0}{\beta\epsilon^3}, \qquad \lambda = \frac{\lambda_0}{\beta\epsilon^3} \tag{1}$$

- For sake of simplicity we assume free boundary conditions along the whole boundary of Ω. The discussion about the different boundary conditions which can be assumed and the way they pass to the limit would make this paper too long. As usual when dealing with Neumann-type boundary conditions, we have to assume that the external forces applied to the structure are balanced and we ensure unicity of equilibrium solution by imposing zero mean rigid motion.

The paper is organized as follows. In Section 2, we first describe precisely the geometry we are interested in by introducing in a sparce way the graphs on which our 3D structures are based. Assuming a plane strain state we state the elastic problem in a 2D domain which corresponds to a thickened graph. Then, in Section 3, we prove that this elastic problem has the same Γ-limit as an equivalent discrete problem set on the nodes of the graph. Both extensional and flexional stiffnesses must be taken into account even if the flexional rigidity is much lower than the extensional one. This part is rather technical and the proofs which are more or less standard are postponed to the Appendix.

In Section 4 we attack the problem of finding the Γ-limit of the discrete energy. We study the problem from the variational point of view adapting to our case the tools of Γ-convergence, [15], [10] and double-scale limit, [30], [4] which have shown their efficiency for treating many different problems of homogenization. The topology we use is rather weak but it is sufficient to ensure at least, that the equilibrium of the structure under the action forces applied at the nodes of the structure will be well described by the equilibrium of the limit model. We first focus on the pure extensional energy and prove that, for any motion with finite energy, the strain must remain in a particular subspace : uniform such strains must be possible without extending any link of the graph. Then we can compute the energy for non uniform such strains : one part of the energy is due to the flexional rigidity of the links, the other one is due to the non uniformity of the strain and the extensional stiffnesses of the links. Our result differ from the ones given in Ref. [23] or Ref. [24] where the studied discrete system is very similar to our. The point is that, generally, the order of magnitude of the different types of interaction are supposed not to interfere with the homogenization asymtotic process (see Remark 7.5 of Ref. [23], Remark (2.7) of Ref. [26] or Ref. [11]).

In this paper we do not exhaust all interesting questions about our structures : it has been shown in [35] that the types of actions (external forces, external distributions, boundary distributions,. . .) which can be applied to second gradient materials were much richer than the boundary conditions and external forces considered here. However the case we study is sufficient to enlight the way second gradient effects can arise through the homogenization procedure.

2 Initial problem, description of the geometry

2.1 The graph

The geometry we consider is based on a periodic planar graph. Such a graph is determined by

- a prototype cell Y containing a finite number K of nodes the position of which is denoted y_s, $s \in \{1, \dots, K\}$;
- two independent periodicity vectors t_1, t_2. Introducing, for $I = (i, j) \in \mathbb{N}^2$, the points $y_{I,s} := y_s + it_1 + jt_2$, the set of nodes of the graph is

$$\mathring{G} := \left\{ y_{I,s} \ : \ I \in \mathbb{N}^2,\ s \in \{1, \dots, K\} \right\}$$

  We use $y_I := \frac{1}{K} \sum_{s=1}^{K} y_{I,s}$ as a reference point in the cell I. As the graph will be re-scaled, we can assume without loss of generality that $t_1 \times t_2 = 1$ (i.e. the area $|\mathrm{Y}| = 1$);

- five $K \times K$ matrices a_p taking value in $\mathbb{R}^+$ defining the edges of the graph : an edge links nodes $y_{I,s}$ and $y_{I+p,s'}$ as soon as $a_{p,s,s'} > 0$. Here p belongs to the set[1]

$$\mathcal{P} := \{(0, 0), (1, 0), (0, 1), (1, 1), (1, -1)\}.$$

  We denote $\mathbf{p} := p_1 t_1 + p_2 t_2 \in \{0, t_1, t_2, t_1 + t_2, t_1 - t_2\}$ the corresponding vector so that $y_{I+p,s} = y_{I,s} + \mathbf{p}$. We introduce the set of multi-indices corresponding to all edges :

$$\mathcal{A} := \{(p, s, s') : \ p \in \mathcal{P},\ 1 \le s \le K,\ 1 \le s' \le K,\ a_{p,s,s'} > 0\}.$$

[1] Note that, owing to periodicity, only half of the neighbors of a cell have been considered. Note also that there is no loss of generality (as soon as we assume the range of interactions is finite) in assuming that a cell is interacting only with its closest neighbors. Indeed we can always choose a prototype cell large enough for this assumption to become true.

To resume the graph reads

$$G := \bigcup_{I\in\mathbb{N}^2} \bigcup_{(p,s,s')\in\mathcal{A}} [y_{I,s}, y_{I+p,s'}]$$

Restrictive assumptions : Not all interaction matrices are admissible :

- There is no crossing or overlapping of different edges : for any (p, s, s') and $(\tilde{p}, \tilde{s}, \tilde{s}')$ in $\mathcal{A}$,

$$[y_{I,s}, y_{I+p,s'}] \cap [y_{\tilde{I},\tilde{s}}, y_{\tilde{I}+\tilde{p},\tilde{s}'}] \not\subset \{y_{I,s}, y_{I+p,s'}\} \Rightarrow (\tilde{I}, \tilde{s}, \tilde{p}) = (I, s, p).$$

  This assumption results from the cylindrical shapes we are studying but is not fundamental. One could design multilayered structures, allowing crossing of interactions. The reduction to a discrete problem would then have to be adapted to this case.

- We are not interested by lattices which are made of several disconnected lattices. So we assume that the edges connect all the nodes of the structures. More precisely we assume that, for any $p \in \mathcal{P}$ and any $(s, s') \in \{1, \ldots, K\}^2$, there exist finite sequences $(s_1, \ldots, s_{r+1})$ in $\{1, \ldots, K\}$, $(p_1, \ldots, p_r)$ in $\mathcal{P}$, $(\epsilon_1, \ldots, \epsilon_r)$ in $\{-1, 1\}$ such that $s_1 = s$, $s_{r+1} = s'$, $\sum_{i=1}^r \epsilon_i \mathbf{p}_i = \mathbf{p}$,

$$\epsilon_i > 0 \Rightarrow (p_i, s_i, s_{i+1}) \in \mathcal{A} \qquad \text{and} \qquad \epsilon_i < 0 \Rightarrow (p_i, s_{i+1}, s_i) \in \mathcal{A}.$$

  Let us denote $\mathcal{P}_R$ the set $\{(i, j),\ max(|i|, |j|) \leq R\}$. The previous condition implies the existence of some integer R such that, for any I, $J \in \mathcal{P}_R$, $(s, s') \in \{1, \ldots, K\}^2$ and any square summable quantity φ,

$$(\forall I' \in I + \mathcal{P}_{2R},\ \forall (p, t, t') \in \mathcal{A},\ \varphi_{I'+p,t'} - \varphi_{I',t} = 0) \Longrightarrow (\varphi_{I+J,s'} - \varphi_{I,s} = 0).$$

  This implies that the quantity $\sum_{(I',p,t,t')\in\mathcal{P}_{2R}\times\mathcal{A}} a_{p,t,t'}(\varphi_{I'+p,t'} - \varphi_{I',t})^2$ controls $(\varphi_{I+J,s'} - \varphi_{I,s})^2$ and we get the existence of a constant C such that for any $J \in \mathcal{P}_R$ and $(s, s') \in \{1, \ldots, K\}^2$,

$$\sum_{I\in\mathbb{N}^2} (\varphi_{I+J,s'} - \varphi_{I,s})^2 \leq C \sum_{(I,p,t,t')\in\mathbb{N}^2\times\mathcal{A}} a_{p,t,t'}(\varphi_{I+p,t'} - \varphi_{I,t})^2 \tag{2}$$

- The previous connectedness assumption is necessary but not sufficient for the structure to be able to resist to reasonable external forces. Before formulating the new assumption, let us introduce some supplementary notation. For any edge $[y_{I,s}, y_{I+p,s'}]$ of the considered graph, we denote

$$\ell_{p,s,s'} := \|y_{I+p,s'} - y_{I,s}\|, \quad \tau_{p,s,s'} := (y_{I+p,s'} - y_{I,s})/\ell_{p,s,s'}$$

  its length[2] and direction and $\tau^{\perp}_{p,s,s'}$ which completes $\tau_{p,s,s'}$ in a direct orthonormal basis. For any vector field vector field $U_{I,s}$ defined on $\mathbb{N}^2 \times \{0, \ldots, K\}$, we consider the associated *extension* and *rotation* defined on each edge of the graph by

$$(\rho_U)_{I,p,s,s'} := (U_{I+p,s'} - U_{I,s}) \cdot \tau_{p,s,s'},$$
$$(\alpha_U)_{I,p,s,s'} := \frac{1}{l_{p,s,s'}}(U_{I+p,s'} - U_{I,s}) \cdot \tau^{\perp}_{p,s,s'}$$

  We say that an edge $(p, s, t) \in \mathcal{A}$ is *controlled in the direction* $J \in \mathcal{P}_R$ *by extension* if there exists a constant C such that, for any vector field U,

$$\sum_{I\in\mathbb{N}^2} \Big((\alpha_U)_{I+J,p,s,t} - (\alpha_U)_{I,p,s,t}\Big)^2 \leq C \sum_{(I',p',s',t')\in\mathbb{N}^2\times\mathcal{A}} a_{p',s',t'}\Big((\rho_U)_{I',p',s',t'}\Big)^2 \tag{3}$$

  We assume that there exists in the graph at least two edges (p, s, s') and (q, t, t') which are controlled in independent directions J and J'. In section 4.1 we will show that this assumption together with the connectedness assumption ensures the relative compactness of sequences with finite energy.

Some examples of graphs satisfying these assumptions are given in figures 3 and 4 while the structures of figure 2 are not stiff enough either because they are not connected (fig 2b) or they do not satisfy the last assumption above.

[2]We also denote ℓ_m the minimal value of all lengths $\ell_{p,s,s'}$: $\ell_m := \min\{\ell_{p,s,s'}\ :\ (p, s, s') \in \mathcal{A}\}$.

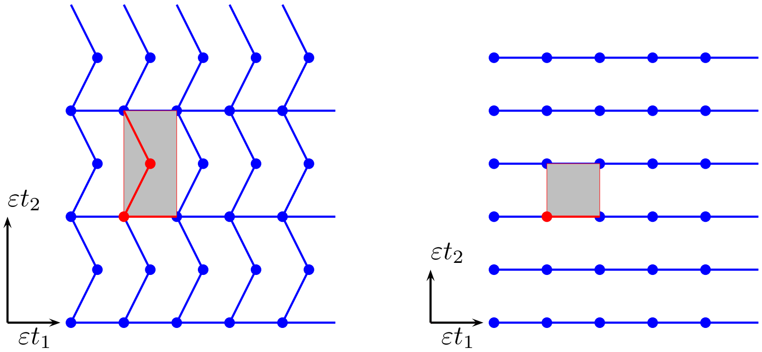

Figure 2: Structures for which relative compactness is not ensured.

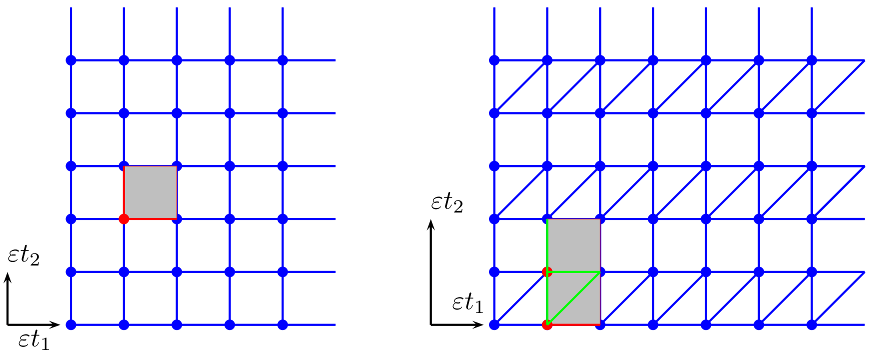

Figure 3: Admissible structures.

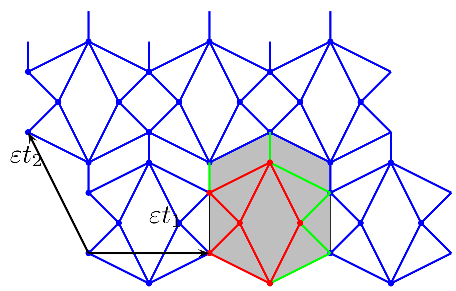

Figure 4: Another admissible structure : the pantographic one.

2.2 The 2D elastic problem

We assume without loss of generality that $L = 1$ (choice of the unit length). Therefore our assumptions resume in

$$\ell = \varepsilon, \quad e = \beta\varepsilon^2, \quad \mu = \frac{\mu_0}{\beta\epsilon^3}, \quad \lambda = \frac{\lambda_0}{\beta\epsilon^3}.$$

Let Ω be a bounded convex domain in $\mathbb{R}^2$ which, for sake of simplicity, is assumed to be of measure 1. Let $\mathcal{I}^\varepsilon$ be the set of rescaled cells which lie sufficiently inside the domain :

$$\mathcal{I}^\varepsilon := \Big\{ I;\ \forall s\, y_{I,s} \in \Omega \text{ and } d(\varepsilon y_{I,s}, \partial\Omega) > \sqrt{\varepsilon} \Big\}$$

(where d stands for the Euclidian distance) and G^ε the union of the edges which link the nodes of these cells.

$$G^\varepsilon := \bigcup \Big\{ [\varepsilon y_{I,s}, \varepsilon y_{I+p,s'}] :\ (I, p, s, s') \in \mathcal{I}^\varepsilon \times \mathcal{A} \Big\}$$

The number N^ε of such cells is equivalent to ε^{-2} and we will denote in the sequel the mean value of any quantity φ defined on $\mathcal{I}^\varepsilon$ as

$$\sum\!\!\!\!\!\!-_{I} \varphi_I := \frac{1}{N^\varepsilon} \sum_{I\in\mathcal{I}^\varepsilon} \varphi_I \sim \varepsilon^2 \sum_{I\in\mathcal{I}^\varepsilon} \varphi_I$$

The planar elastic problem is set in the thickened graph:

$$\Omega^\varepsilon := \left\{x \in \Omega;\ d(x, G^\varepsilon) < \beta\varepsilon^2\right\}, \tag{4}$$

where the thickened nodes $B_{I,s}$

$$B_{I,s} := \{x;\ d(x, y_{I,s}) < \beta\varepsilon^2\}$$

play an essential role.

The elastic energy $\mathscr{E}_\varepsilon$ is defined, for any displacement field $u \in \mathrm{L}^2(\Omega^\varepsilon, \mathbb{R}^2)$ with zero mean rigid motion, by

$$\mathscr{E}_\varepsilon(u) := \begin{cases} \dfrac{1}{\beta\varepsilon^3} \displaystyle\int_{\Omega^\varepsilon} \left(\mu_0 \|e(u)\|^2 + \frac{\lambda_0}{2} \mathrm{tr}(e(u))^2\right) dx & \text{if } u \in \mathrm{H}^1(G_\varepsilon, \mathbb{R}^2), \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

Here $e(u)$ denotes the symmetric part of the gradient of u ($e(u) = (\nabla u + \nabla^t u)/2$ is the linearized strain tensor), $\mathrm{tr}(e(u))$ denotes the trace of the matrix $e(u)$. To Lamé coefficients which satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 > 0$, we associate Young modulus

$$Y = \frac{Y_0}{\beta\varepsilon^3} \quad \text{where} \quad Y_0 := \frac{4\mu_0(\mu_0 + \lambda_0)}{2\mu_0 + \lambda_0}$$

and Poisson ratio

$$\nu := \frac{\lambda}{2\mu + \lambda} = \nu_0 := \frac{\lambda_0}{2\mu_0 + \lambda_0}.$$

The reader may have noticed that the values of the positive coefficients $a_{p,s,s'} > 0$ of the interaction matrices were, up to now, irrelevant (as soon as they remain positive). We now fix them by setting

$$a_{p,s,s'} = \frac{2Y_0}{\ell_{p,s,s'}}. \tag{6}$$

2.3 Convergence

In order to study the homogenization of the considered structures, we need to specify the way we pass to the limit of a sequence of fields (u^ε) with finite energy $\mathscr{E}_\varepsilon(u^\varepsilon) < +\infty$. Indeed each term is defined on a different domain Ω^ε. To that aim, we first introduce the operator $u \to \bar{u}$, which to any field $u \in \mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$ associates the family $\bar{u}$ of mean values defined for $I \in \mathcal{I}^\varepsilon$ and $s \in \{1, \ldots, K\}$ by

$$\bar{u}_{I,s} := \fint_{B_{I,s}} u(x)\, dx := \frac{1}{|B_{I,s}|} \int_{B_{I,s}} u(x)\, dx. \tag{7}$$

Note that this operator which maps $\mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$ onto the set $\mathcal{V}_\varepsilon$ of functions defined on $\mathcal{I}^\varepsilon \times \{1, \ldots, K\}$ actually depends on ε, even if the notation does not recall it.

Then we define the convergence of a sequence of families of vectors $Z^\varepsilon \in \mathcal{V}_\varepsilon$: We say that (Z^ε) converges to the measurable function z, and we write $Z^\varepsilon \rightharpoonup z$, when the following weak* convergence of measures holds true :

$$\sum\!\!\!\!\!\!-_{I} \frac{1}{K} \sum_{s=1}^{K} Z^\varepsilon_{I,s} \delta_{\varepsilon y_{I,s}} \overset{*}{\rightharpoonup} z(x)\, dx \tag{8}$$

where δ_y stands for the Dirac measure at point y. Finally we say that the sequence of functions (u^ε) (where $u^\varepsilon \in \mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$) converges to u when $\bar{u}^\varepsilon \rightharpoonup u$. As no confusion can arise, we simply write $u^\varepsilon \rightharpoonup u$.

Remark 1. *The convergence (8) means that, for all $\varphi \in C^0(\Omega)$,*

$$\sum_I K^{-1} \sum_{s=1}^{K} Z^\varepsilon_{I,s} \varphi(\varepsilon y_{I,s}) \to \int_\Omega z(x)\varphi(x)\,dx \tag{9}$$

When applying this notion of convergence to sequences (Z^ε) such that $\sum_I \|Z^\varepsilon_{I,s}\|^2$ is bounded, we are thus assured that a subsequence converges to some $z \in \mathrm{L}^2(\Omega)$. In view of (9), we note that we can replace in (8) the Dirac measure $\delta_{\varepsilon y_{I,s}}$ by $\delta_{\varepsilon y_I}$ or even $\delta_{\varepsilon y_{I+p}}$. Indeed $\varphi(\varepsilon y_{I+p}, s') - \varphi(\varepsilon y_{I,s}, s) = O(\varepsilon)$.

Remark 2. *The convergence of measures (8) imply that, for any convex function Φ,*

$$\liminf_{\varepsilon} \sum_I \frac{1}{K} \sum_{s=1}^{K} \Phi(Z^\varepsilon_{I,s}) \geq \int_\Omega \Phi(z(x))\,dx. \tag{10}$$

Indeed it is enough to use the fact that Φ is the supremum of a countable set of affine functions.

Remark 3. *The choice of this convergence limits the application of our homogenization result to forces applied at the "nodes" of the structure. More precisely to forces fields f^ε of the type*

$$f^\varepsilon(x) = \frac{1}{K\pi\beta^2\varepsilon^4} \sum_{I\in\mathcal{I}} \sum_{s=1}^{K} f(y_{I,s}) 1_{B_{I,s}}(x)$$

where f is a continuous field. Actually our result can apply to more general forces. Indeed, using Korn inequality on each rectangular edge of the structure, it is possible to show that the convergence $u_\varepsilon \rightharpoonup u$ we just defined is equivalent to the weak convergence of $\frac{|\Omega|}{|\Omega^\varepsilon|} u_\varepsilon 1_{\Omega^\varepsilon}$ to u. Hence forces applied on the whole structure (like weight) are also admissible.*

3 Reduction to a discrete problem

We prove in this section that the considered structure can be studied as a discrete one. To any function (U, θ) defined on the nodes of the graph (U being vector valued while θ is scalar), we associate the energies

$$E_\varepsilon(U) := \sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}} \frac{a^p_{s,s'}}{2} \left(\frac{U_{I+p,s'} - U_{I,s}}{\varepsilon} \cdot \tau_{p,s,s'} \right)^2 \tag{11}$$

$$F_\varepsilon(U,\theta) := \varepsilon^2 \sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}} \frac{a^p_{s,s'}\beta^2}{6} \Big(3\big(\theta_{I+p,s'} + \theta_{I,s} - \frac{2}{\ell_{p,s,s'}} \frac{U_{I+p,s'} - U_{I,s}}{\varepsilon} \cdot \tau^\perp_{p,s,s'}\big)\big)^2 + \left(\theta_{I+p,s'} - \theta_{I,s}\right)^2 \Big). \tag{12}$$

The sum $E_\varepsilon + F_\varepsilon$ corresponds to the elastic energy of a system of nodes linked by extensional and flexional bars. This section is devoted to the proof of the following theorem which states that the Γ-limit of the initial sequence of 2D elastic energies is identical to the limit of the sequence of these discrete energies.

Theorem 1. *The sequences $(\mathscr{E}_\varepsilon)$ and $(E_\varepsilon + F_\varepsilon)$ share the same Γ-limit $\mathscr{E}$. Indeed, for any measurable function u, we have*

(i) $$\inf_{u^\varepsilon} \{\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \;:\; u^\varepsilon \rightharpoonup u\} \geq \inf_{U^\varepsilon,\theta^\varepsilon} \{\liminf_{\varepsilon\to 0} (E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon));\; U^\varepsilon \rightharpoonup u\}$$

(ii) $$\inf_{u^\varepsilon} \{\limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \;:\; u^\varepsilon \rightharpoonup u\} \leq \inf_{U^\varepsilon,\theta^\varepsilon} \{\limsup_{\varepsilon\to 0} (E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon)));\; U^\varepsilon \rightharpoonup u\}$$

We first recall some results concerning the elastic behavior of a thin rectangle which are well known in an asymptotic form but that we need here to state more precisely in order to be able to apply them to the whole structure. Their proofs are postponed to the Appendix.

3.1 Estimations for an elastic rectangle

We use the orthonormal basis (e_1, e_2) in $\mathbb{R}^2$ and consider the rectangle $\omega := [-\ell/2, \ell/2] \times [-e, +e]$ (with $e < \ell/4$).

To any function $u \in \mathrm{H}^1(\omega)$ we associate

$$U(x_1) := \frac{1}{2e} \int_{-e}^{e} u(x_1, x_2)\, dx_2, \quad \theta(x_1) := -\frac{3}{2e^3} \int_{-e}^{e} u_1(x_1, x_2)\, x_2\, dx_2,$$

$$v(x_1) := \frac{3}{4e^3} \int_{-e}^{e} (u_2(x_1, x_2) - U_2(x_1))(e^2 - x_2^2)\, dx_2.$$

and

$$W := \frac{1}{\pi e^2} \int_{B(0,e)} u(x_1, x_2)\, dx_1 dx_2, \quad \phi := \frac{1}{\pi e^2} \int_{B(0,e)} \frac{\partial_1 u_2 - \partial_2 u_1}{2}(x_1, x_2)\, dx_1 dx_2.$$

Lemma 1. *There exists a constant C independent of e such that, for any $u \in \mathrm{H}^1(B(0,e), \mathbb{R}^2)$*

$$\|U(0) - W\|^2 \leq C \int_{B(0,e)} \|e(u)\|^2 dx, \quad \|\theta(0) - \phi\|^2 \leq C e^{-2} \int_{B(0,e)} \|e(u)\|^2 dx,$$

$$\|v(0)\|^2 \leq C \int_{B(0,e)} \|e(u)\|^2 dx.$$

Proof. By rescaling we can reduce to the case $e = 1$. Let us assume by contradiction that there exists a sequence u^n such that $\int_{B(0,e)} \|e(u^n)\|^2 dx$ tends to zero while one of the quantities $\|U^n(0) - W^n\|^2$, $\|v^n(0)\|^2$ and $\|\theta^n(0) - \phi^n\|^2$ do not tend to zero. Adding if needed a rigid motion to u^n, we can assume $W^n = 0$ and $\phi^n = 0$. From Korn and Poincaré-Wirtingen inequalities we know that $\|u^n\|_{\mathrm{H}^1(B(0,e), \mathbb{R}^2)}$ tends to zero. A trace theorem ensures that u^n tends to zero in $\mathrm{H}^{1/2}(\{0\} \times [-e, e], \mathbb{R}^2)$ and thus in $\mathrm{L}^2(\{0\} \times [-e, e], \mathbb{R}^2)$. In consequence, contrarily to what we have assumed, $U^n(0)$, $v^n(0)$ and $\theta^n(0)$ tend to zero. □

Now, let $0 \leq k < 1 < k' < \ell/(2e)$. In ω, we consider the piecewise constant functions $(\tilde{\mu}, \tilde{\lambda})$ defined by $\tilde{\mu}(x_1, x_2) = \mu$, $\tilde{\lambda}(x) = \lambda$ if $|x_1| < \ell/2 - k'e$, $\tilde{\mu}(x) = k\mu$, $\tilde{\lambda}(x) = k\lambda$, otherwise and we denote respectively (U^-, θ^-, v^-) and (U^+, θ^+, v^+) the values of (U, θ, v) at $x_1 = -\frac{\ell}{2}$ and $x_1 = +\frac{\ell}{2}$.

Lemma 2. *There exists a constant C depending only on k, k' and ν such that, for any $u \in \mathrm{H}^1(\omega)$,*

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \geq \frac{Ye}{\ell} \big(1 - C\frac{e}{\ell}\big) \times \Big[(U_1^+ - U_1^-)^2$$
$$+ \frac{e^2}{3} \Big(3(\theta^+ + \theta^- - 2\frac{U_2^+ - U_2^-}{\ell})^2 + (\theta^+ - \theta^-)^2 \Big) - \frac{e}{2\ell}(v^+ - v^-)^2 \Big]$$

Lemma 3. *There exists a constant C depending only on k, k' and ν such that, for any U^+, U^- in $\mathbb{R}^2$ and θ^+, θ^- in $\mathbb{R}$ there exists $u \in \mathrm{H}^1(\omega, \mathbb{R}^2)$ satisfying $u(x_1, x_2) = U^- + \theta^-(-x_2, x_1)$ if $x_1 < -\frac{\ell}{2} + k'e$, $u(x_1, x_2) = U^+ + \theta^+(-x_2, x_1)$ if $x_1 > \frac{\ell}{2} - k'e$ and*

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \leq \frac{Ye}{\ell} \big(1 + C\frac{e}{\ell}\big)$$
$$\times \Big[(U_1^+ - U_1^-)^2 + \frac{e^2}{3} \Big(3(\theta^+ + \theta^- - 2\frac{U_2^+ - U_2^-}{\ell})^2 + (\theta^+ - \theta^-)^2 \Big) \Big]$$

Proofs of these two lemmas are given in the Appendix.

3.2 Estimation for the whole structure

We can now prove Theorem 1.

Proof. We first notice that the number of edges which concur at a node $y_{I,s}$ of the graph is bounded by $9K$. We set $k = (9K)^{-1}$. Therefore there exists a uniform lowerbound $\theta_m > 0$ for the angles between these different edges. The thickened edges of Ω^ε concurring at node $y_{I,s}$ do not intersect out of the disk of center $y_{I,s}$ and radius $k'e$ with

$k' = (\sin(\theta_m/2))^{-1}$. We consider on Ω^ε the functions $(\tilde\mu, \tilde\lambda)$ defined by $\tilde\mu(x) := \mu_0$, $\tilde\lambda(x) := \lambda_0$, if $d(x, \varepsilon\mathring{G}) > k'\beta\varepsilon^2$, $\tilde\mu(x) = k\mu_0$, $\tilde\lambda(x) = k\lambda_0$, otherwise.

Let u^ε be any sequence of displacement fields with bounded elastic energy $\mathscr{E}_\varepsilon(u^\varepsilon) \le M$ and converging to some function u. Our choice for k and k' allows us to split the energy :

$$\begin{aligned}\mathscr{E}_\varepsilon(u^\varepsilon) &= \frac{1}{\beta\varepsilon^3}\int_{\Omega^\varepsilon}\left(\mu_0\|e(u^\varepsilon)\|^2 + \frac{\lambda_0}{2}\mathrm{tr}(e(u^\varepsilon))^2\right)dx\\ &\ge \frac{1}{\beta\varepsilon^3}\sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}}\int_{S_{I,p,s,s'}}\left(\tilde\mu\|e(u^\varepsilon)\|^2 + \frac{\tilde\lambda}{2}\mathrm{tr}(e(u^\varepsilon))^2\right)dx.\end{aligned}$$

where $S_{I,p,s,s'}$ denotes the rectangle with mean line $[y_{I,s}, y_{I+p,s'}]$ and thickness $2\beta\varepsilon^2$.

Applying Lemma 2 to each term of this sum, we get

$$\begin{aligned}\mathscr{E}_\varepsilon(u^\varepsilon) \ge& \frac{1}{2\varepsilon^2}\big(1 - \frac{C\beta}{\ell_m}\varepsilon\big)\sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}} a_{p,s,s'}\Big[((U^{\varepsilon+}_{I,p,s,s'} - U^{\varepsilon-}_{I,p,s,s'})\cdot\tau_{p,s,s'})^2\\ &- \frac{\beta\varepsilon}{2\ell_{p,s,s'}}(v^{\varepsilon+}_{I,p,s,s'} - v^{\varepsilon-}_{I,p,s,s'})^2 + \frac{\beta^2\varepsilon^2}{3}\Big(3\big(\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} + \theta^{\varepsilon-}_{I,p,s,s'})\\ &- \frac{(U^{\varepsilon+}_{I,p,s,s'} - U^{\varepsilon-}_{I,p,s,s'})\cdot\tau^\perp_{p,s,s'}}{\ell_{p,s,s'}}\big)^2 + (\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} - \theta^{\varepsilon-}_{I,p,s,s'}))^2\Big)\Big]\end{aligned}$$

where $U^{\varepsilon+}_{I,p,s,s'}$, $U^{\varepsilon-}_{I,p,s,s'}$, $v^{\varepsilon+}_{I,p,s,s'}$, $v^{\varepsilon-}_{I,p,s,s'}$, $\theta^{\varepsilon+}_{I,p,s,s'}$, $\theta^{\varepsilon-}_{I,p,s,s'}$ are the quantities associated to u^ε on the rectangle $S_{I,p,s,s'}$ as in Lemma 2.

On the other hand, Lemma 1 states that, for any (p,s,s'), the quantities $\sum_I \|U^{\varepsilon-}_{I,p,s,s'} - \bar u^\varepsilon_{I,s}\|^2$, $\sum_I \|U^{\varepsilon+}_{I,p,s,s'} - \bar u^\varepsilon_{I+p,s}\|^2$, $\sum_I |v^{\varepsilon-}_{I,p,s,s'}|^2$, $\sum_I |v^{\varepsilon+}_{I,p,s,s'}|^2$, $\sum_I |\varepsilon(\theta^{\varepsilon-}_{I,p,s,s'} - \phi^\varepsilon_{I,s})|^2$ and $\sum_I |\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} - \phi^\varepsilon_{I+p,s})|^2$ are all bounded by $\sum_I \int_{B_{I,s}} \|e(u)\|^2$ and thus by $C\varepsilon^3$ with $C = \frac{M\beta}{\min(\mu_0,\mu_0+\lambda_0)}$. Here $\phi^\varepsilon_{I,s}$ is the quantity associated to u^ε on the disk $B_{I,s}$ as in Lemma 1. Hence

$$\begin{aligned}\mathscr{E}_\varepsilon(u^\varepsilon) &\ge \frac{1}{2\varepsilon^2}\sum_{(I,p,s,s')\in\mathcal{A}} a_{p,s,s'}\Big[((\bar u^\varepsilon_{I+p,s'} - \bar u^\varepsilon_{I,s})\cdot\tau_{p,s,s'})^2 + \frac{\beta^2\varepsilon^2}{3}\Big(3(\varepsilon(\phi^\varepsilon_{I+p,s'} + \phi^\varepsilon_{I,s})\\ &\quad - \frac{(\bar u^\varepsilon_{I+p,s'} - \bar u^\varepsilon_{I,s})\cdot\tau^\perp_{p,s,s'}}{\ell_{p,s,s'}})^2 + (\varepsilon(\phi^\varepsilon_{I+p,s'} - \phi^\varepsilon_{I,s}))^2\Big)\Big] + O(\sqrt{\varepsilon})\\ &\ge E_\varepsilon(\bar u^\varepsilon) + F_\varepsilon(\bar u^\varepsilon, \phi^\varepsilon) + O(\sqrt{\varepsilon})\end{aligned}$$

Passing to the limit we get

$$\begin{aligned}\liminf \mathscr{E}_\varepsilon(u^\varepsilon) &\ge \liminf\big(E_\varepsilon(\bar u^\varepsilon) + F_\varepsilon(\bar u^\varepsilon, \phi^\varepsilon)\big)\\ &\ge \inf_{U^\varepsilon,\theta^\varepsilon}\Big\{\liminf\big(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon)\big);\ U^\varepsilon \rightharpoonup u\Big\}\end{aligned}$$

This being true for any sequence (u^ε) converging to some function u with bounded energy, point (i) is proven.

Now let u be a measurable vector valued function and consider any sequence $(U^\varepsilon, \theta^\varepsilon)$ with bounded energy $(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) < M)$ and such that $U^\varepsilon \rightharpoonup u$.

On each thickened edge $S_{I,p,s,s'}$, Lemma 3 provides a piecewise C^1 function $u^\varepsilon_{I,p,s,s'}$ satisfying

$$u^\varepsilon_{I,p,s,s'}(x_1,x_2) = U^\varepsilon_{I,s} + \theta^\varepsilon_{I,s}\times(-x_2,x_1) \text{ on } B_{I,s}$$

$$u^\varepsilon_{I,p,s,s'}(x_1,x_2) = U^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I+p,s'}\times(-x_2,x_1) \text{ on } B_{I+p,s'}$$

and such that

$$\begin{aligned}\int_{S_{I,p,s,s'}}\left(\mu\|e(u^\varepsilon_{I,p,s,s'})\|^2 + \frac{\lambda}{2}\mathrm{tr}(e(u^\varepsilon_{I,p,s,s'}))^2\right)dx \le \frac{a_{p,s,s'}}{2\varepsilon^2}(1 + \frac{C\beta}{l_{p,s,s'}}\varepsilon)\\ \Bigg(\Big((U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s})\cdot\tau_{p,s,s'}\Big)^2 + \frac{\beta^2\varepsilon^2}{3}\Big(3(\varepsilon\theta^\varepsilon_{I+p,s'} + \varepsilon\theta^\varepsilon_{I,s})\\ + \frac{2}{l_{p,s,s'}}((U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s})\cdot\tau^\perp_{p,s,s'})\Big)^2 + (\varepsilon\theta^\varepsilon_{I+p,s'} - \varepsilon\theta^\varepsilon_{I,s})^2\Bigg).\end{aligned}$$

We can now define u^ε on Ω_ε by setting $u^\varepsilon(x) := u^\varepsilon_{I,p,s,s'}(x)$ if $x \in S_{I,p,s,s'}$. Our assumptions about the geometry of the graph and our definition of k' make this definition coherent on the intersections of different edges. By definition $\bar{u}^\varepsilon = U^\varepsilon$ and so $u^\varepsilon \rightharpoonup u$. By summation we get

$$\mathscr{E}_\varepsilon(u^\varepsilon) \leq (1 + \frac{C\beta}{l_m}\varepsilon)(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)))$$

Passing to the limit

$$\inf_{u^\varepsilon \rightharpoonup u} \limsup \mathscr{E}_\varepsilon(u^\varepsilon) \leq \limsup \mathscr{E}_\varepsilon(u^\varepsilon) \leq \limsup(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon))).$$

This being true for any sequence (θ^ε) and any sequence (U^ε) converging to u, Point (ii) is proven. □

4 Main results

From now on we will seek for the Γ-limit $\mathscr{E}$ of the sequence of the discrete functionals $(E_\varepsilon + F_\varepsilon)$ defined in (11), (12).

We do not intend to study the way the different boundary conditions which could be imposed to our structures pass to the limit. That is a very interesting topic as the boundary conditions associated to second gradient material are rich and have exotic mechanical interpretation,[16],[35]. But studying their whole diversity would lead to very long mathematical developments. On the other hand, as the structures we consider may present in the limit some inextensibility constraint, assuming, at it is frequent, Dirichlet boundary conditions would lead to a trivial set of admissible deformations. So we decide to consider here only free boundary conditions. As well known, in this case, the equilibrium of the structure can be reached only when the applied external actions are balanced and the solution of equilibrium problems is defined up to a global rigid motion. In order to ensure uniqueness, we need to impose that U and θ have zero mean values:

$$\sum_I \frac{1}{K}\sum_{s=1}^{K} U_{I,s} = 0, \quad \sum_I \frac{1}{K}\sum_{s=1}^{K} \theta_{I,s} = 0 \tag{13}$$

4.1 Compactness

Let us consider a sequence $(U^\varepsilon, \theta^\varepsilon)$ with bounded energy : $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) \leq M$. We associate to this sequence the families of vectors m^ε_I, $v^\varepsilon_{I,s}$ and $\chi^\varepsilon_{I,p}$ defined by

$$m^\varepsilon_I := \frac{1}{K}\sum_{s=1}^{K} U^\varepsilon_{I,s}, \quad v^\varepsilon_{I,s} := \frac{1}{\varepsilon}(U^\varepsilon_{I,s} - m^\varepsilon_I), \quad \chi^\varepsilon_{p,I} := \frac{1}{\varepsilon}(m^\varepsilon_{I+p} - m^\varepsilon_I). \tag{14}$$

and the family of reals $\omega^\varepsilon_{I,p,s,s'}$ defined by

$$\omega^\varepsilon_{I,p,s,s'} := \begin{cases} \varepsilon^{-2}(U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \cdot \tau_{p,s,s'}, & \text{if } (p,s,s') \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Lemma 4. *Let $(U^\varepsilon, \theta^\varepsilon)$ satisfying $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) \leq M$, then the sequences $(\sum_I \|U^\varepsilon_{I,s}\|^2)$, $(\sum_I (\theta^\varepsilon_{I,s})^2)$, $(\sum_I \|m^\varepsilon_I\|^2)$, $(\sum_I \|v^\varepsilon_{I,s}\|^2)$, $(\sum_I \|\chi^\varepsilon_{I,p}\|^2)$ and $(\sum_I (\omega^\varepsilon_{I,p,s,s'})^2)$ are bounded.*

Proof. Let us also associate to U^ε the quantities

$$(\rho^\varepsilon)_{I,p,s,s'} := \frac{U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}}{\varepsilon} \cdot \tau_{p,s,s'}, \quad (\alpha^\varepsilon)_{I,p,s,s'} := \frac{U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}}{\varepsilon \ell_{p,s,s'}} \cdot \tau^\perp_{p,s,s'}, \tag{16}$$

The bound for the energy implies[3]

$$\sum_I \sum_{(p,s,s') \in \mathcal{A}} a_{p,s,s'} \left(\rho^\varepsilon_{I,p,s,s'}\right)^2 \leq M\varepsilon^2 \tag{17}$$

$$\sum_I \sum_{(p,s,s') \in \mathcal{A}} a_{p,s,s'} \left(3\big(\theta^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I,s} - 2\alpha^\varepsilon_{I,p,s,s'}\big)^2 + (\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s})^2\right) \leq M \tag{18}$$

[3] M is a bound which varies from line to line.

Inequality (17) gives directly the desired bound for $\sum_I (\omega^\varepsilon_{I,p,s,s'})^2$. From inequality (18) we get

$$\sum_I \sum_{(p,s,s')\in\mathcal{A}} a_{p,s,s'} (\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s})^2 \le M.$$

This has two consequences. First, owing to the connectedness assumption (2),

$$\forall J \in \mathcal{P}_R,\ \forall (s,s') \in \{1,\dots,K\}^2,\ \sum_I (\theta^\varepsilon_{I+J,s'} - \theta^\varepsilon_{I,s})^2 \le M. \tag{19}$$

Then, using again (18),

$$\sum_I \sum_{(p,s,s')\in\mathcal{A}} a_{p,s,s'} \left(\theta^\varepsilon_{I,s} - \alpha^\varepsilon_{I,p,s,s'}\right)^2 \le M. \tag{20}$$

Consider now the two edges (p,s,t) and (p',s',t') which we have assumed to be controlled in independent directions J and J' following definition (3). Inequality (17) leads to

$$\sum_I \left(\alpha^\varepsilon_{I+J,p,s,t} - \alpha^\varepsilon_{I,p,s,t}\right)^2 \le C \sum_I \sum_{(p',s',t')\in\mathcal{A}} a_{p',s',t'} \left(\rho^\varepsilon_{I',p',s',t'}\right)^2 \le CM\varepsilon^2.$$

By triangle inequality we get, for any $k \in \mathbb{Z}$ such that $I + I' \in \mathcal{I}^\varepsilon$ the bound $\sum_I \left(\alpha^\varepsilon_{I+kJ,p,s,t} - \alpha^\varepsilon_{I,p,s,t}\right)^2 \le M$. Then, using (20) and (19),

$$\sum_I \left(\theta^\varepsilon_{I+kJ,s'} - \theta^\varepsilon_{I,s}\right)^2 \le M.$$

Combining this result with the similar result obtained considering the second edge (p',s',t') controlled in the direction J' we get, for any $(k,k') \in \mathbb{Z}^2$,

$$\sum_I \left(\theta^\varepsilon_{I+kJ+k'J',s'} - \theta^\varepsilon_{I,s}\right)^2 \le M.$$

Noticing that, for any $I' \in \mathbb{Z}^2$, there exist $(k,k') \in \mathbb{Z}^2$ and $J'' \in \mathcal{P}_R$ such that $I' = I + kJ + k'J' + J''$, we obtain using once again (19), $\sum_I \left(\theta^\varepsilon_{I+I',s'} - \theta^\varepsilon_{I,s}\right)^2 \le M$. Taking the mean value with respect to I' and s' and using (13), we get the desired bound for the field θ :

$$\sum_I \left(\theta^\varepsilon_{I,s}\right)^2 \le M. \tag{21}$$

From (20) and (21) we directly have

$$\sum_I a_{p,s,s'} \left(\alpha^\varepsilon_{I,p,s,s'}\right)^2 \le M$$

which combined with (17) gives

$$\sum_I a_{p,s,s'} \left(\frac{\|U_{I+p,s'} - U_{I,s}\|}{\varepsilon}\right)^2 \le M.$$

We use again the connectedness assumption (2) and get, for any $p \in \mathcal{P}$ and $(s,s') \in \{1,\dots,K\}^2$,

$$\sum_I \left(\frac{\|U_{I+p,s'} - U_{I,s}\|}{\varepsilon}\right)^2 \le M. \tag{22}$$

By triangle inequality, for any $I' \in \mathbb{Z}^2$ and $(s,s') \in \{1,\dots,K\}^2$, one gets $\sum_I (\|U_{I+I',s'} - U_{I,s}\|)^2 \le M$. Taking the mean value with respect to I' and s' and using (13), we finally obtain

$$\sum_I \|U_{I,s}\|^2 \le M. \tag{23}$$

Other bounds are now easy to get : taking the mean value with respect to s in (23) and (22) (with $p = 0$) gives respectively

$$\sum_I \|m_I\|^2 \le M, \quad \sum_I \|v_{I,s}\|^2 \le M, \tag{24}$$

while taking the mean value with respect to s and s' in (22) gives

$$\sum_I \|\chi_{I,p}\|^2 \le M. \tag{25}$$

□

4.2 Double scale convergence

Let us now introduce a finer notion of convergence than (8) : we say that the sequence (Z^e) double-scale converges to the measurable function z defined on $\mathbb{R}^2 \times \mathcal{I}^\varepsilon$ and we write $Z^\varepsilon \rightharpoonup\!\!\rightharpoonup z$, if the following weak* convergences of measures hold true :

$$\forall s \in \{1, \dots, K\}, \quad \sum_{I \in \mathcal{I}} Z^\varepsilon_{I,s} \delta_{\varepsilon y_{I,s}} \overset{*}{\rightharpoonup} z(x, s)\, dx \tag{26}$$

Note that we write indifferently $z(x, s)$ or $z_s(x)$. Even if this definition is nothing else than a weak convergence for all fixed s, it is enlightening to see this definition as a generalization of the double-scale convergence defined in Ref. [30] or Ref. [4]. Indeed (8) is equivalent to the fact that, for all $\varphi \in C^0(\Omega \times \{1, \dots, K\})$,

$$\sum_I \sum_{s=1}^K Z^\varepsilon_{I,s} \varphi(\varepsilon y_{I,s}, s) \to \int_\Omega \sum_{s=1}^K z(x, s) \varphi(x, s)\, dx \tag{27}$$

Note that Remark 1 still apply : if $\sum_I \|Z^\varepsilon_{I,s}\|^2$ is bounded, we are thus assured that some subsequence double-scale converges. We also can replace in (26) the Dirac measure $\delta_{\varepsilon y_{I,s}}$ by $\delta_{\varepsilon y_I}$ or by $\delta_{\varepsilon y_{I+p}}$. In consequence this notion of convergence can also be applied to sequences of functions (Z^ε) that do not explicitly depend on s (like m^ε or χ^ε_p) : in that case it is equivalent to write $Z^\varepsilon \rightharpoonup\!\!\rightharpoonup z$ or $Z^\varepsilon \rightharpoonup z$.

The bounds that we established in Lemma 4 imply the existence of θ, u, v, χ_p and $\omega_{p,s,s'}$ in L^2 such that, up to subsequences,

$$\theta^\varepsilon \rightharpoonup\!\!\rightharpoonup \theta, \quad m^\varepsilon \rightharpoonup u, \quad v^\varepsilon \rightharpoonup\!\!\rightharpoonup v, \quad \chi^\varepsilon_p \rightharpoonup \chi_p \quad \text{and} \quad \omega^\varepsilon_{p,s,s'} \rightharpoonup \omega_{p,s,s'}. \tag{28}$$

We also have

$$U^\varepsilon \rightharpoonup\!\!\rightharpoonup u, \qquad \sum_{s=1}^K v(x, s) = 0 \quad \text{and} \qquad \chi_p = \nabla u \cdot \mathbf{p}. \tag{29}$$

Indeed, the convergence of v^ε implies that $(U^\varepsilon - m^\varepsilon) \rightharpoonup\!\!\rightharpoonup 0$ and so $U^\varepsilon \rightharpoonup\!\!\rightharpoonup u$; the fact that $\sum_{s=1}^K v^\varepsilon_{I,s} = 0$ clearly implies that $\sum_{s=1}^K v_s(x) = 0$. To check that $\chi_p = \nabla u \cdot \mathbf{p}$, it is enough to notice that, for any smooth test field φ

$$\begin{aligned}
\int_\Omega \chi_p(x) \cdot \varphi(x) &= \lim \sum_I \varepsilon^{-1}(m^\varepsilon_{I+p} - m^\varepsilon_I) \cdot \varphi(\varepsilon y_I) \\
&= \lim \sum_I m^\varepsilon_I \cdot \varepsilon^{-1}(\varphi(\varepsilon y_{I-p}) - \varphi(\varepsilon y_I)) = \lim \sum_I m^\varepsilon_I \cdot (-\nabla \varphi(\varepsilon y_I) \cdot \mathbf{p}) + O(\varepsilon) \\
&= -\int_\Omega u(x) \cdot (\nabla \varphi(x) \cdot \mathbf{p}) = \int_\Omega (\nabla u(x) \cdot \mathbf{p}) \cdot \varphi(x).
\end{aligned}$$

4.3 Homogenization result

Using the notation introduced in the previous sections, we can rewrite the two addends of the energy, $E_\varepsilon(U^\varepsilon)$ and $F_\varepsilon(U^\varepsilon, \theta^\varepsilon)$, under the forms

$$\bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) := \varepsilon^{-2} \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2} \big((v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s} + \chi^\varepsilon_{I,p}) \cdot \tau_{p,s,s'} \big)^2 \tag{30}$$

$$\begin{aligned}
\bar{F}_\varepsilon(v^\varepsilon, \chi^\varepsilon, \theta^\varepsilon) := \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'} \beta^2}{6} \Big(& 3\big(\theta^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I,s} \\
& - \frac{2}{\ell_{p,s,s'}} (v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s} + \chi^\varepsilon_{I,p}) \cdot \tau^\perp_{p,s,s'} \big)^2 + \big(\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s}\big)^2 \Big)
\end{aligned} \tag{31}$$

Let us introduce the continuous counterparts of these quantities. For functions θ, v defined respectively on $\Omega \times \{1, \dots, K\}$ and η defined on $\Omega \times \mathcal{P} \times \{1, \dots, K\}$, square integrable with respect to their first variable and taking value respectively in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^2$, we set

$$\bar{E}(v,\eta) := \int_\Omega \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}((v_{s'}(x) - v_s(x) + \eta_{p,s'}(x)) \cdot \tau_{p,s,s'})^2 \, dx, \tag{32}$$

$$\begin{aligned}\bar{F}(v,\eta,\theta) := \int_\Omega \sum_{p,s,s'} \frac{a_{p,s,s'}\beta^2}{6} \Big(3\big(\theta_{s'}(x) + \theta_s(x) \\ - \frac{2}{\ell_{p,s,s'}}(v_{s'}(x) - v_s(x) + \eta_{p,s'}(x)) \cdot \tau^\perp_{p,s,s'}\big)^2 + \big(\theta_{s'}(x) - \theta_s(x)\big)^2 \Big).\end{aligned} \tag{33}$$

We extend this definition to distributions by setting $\bar{E} = +\infty$ or $\bar{F} = +\infty$ whenever the quantities are not square integrable.

For any functions u and v respectively in $\mathrm{L}^2(\mathbb{R}^2)$ and $\mathrm{L}^2(\mathbb{R}^2 \times \{1,\dots,K\})$ we set, in the sense of distributions, for any $(p,s) \in \mathcal{P} \times \{1,\dots,K\}$

$$(\eta_u)_{p,s} := \nabla u \cdot \mathbf{p}. \tag{34}$$

$$(\xi_{u,v})_{p,s} = \nabla v_s \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u \cdot \mathbf{p} \cdot \mathbf{p}. \tag{35}$$

The limit energy of our structure reads

$$\mathscr{E}(u) := \inf_{w,v,\theta} \left\{ \bar{E}(w, \xi_{u,v}) + \bar{F}(v, \eta_u, \theta);\ \bar{E}(v,\eta_u) = 0 \right\}. \tag{36}$$

when all integrands are integrable functions, $+\infty$ otherwise. Indeed we have

Theorem 2. *The sequence $(E_\varepsilon + F_\varepsilon)$ Γ-converges to $\mathscr{E}$:*
(i) For all sequence $(U^\varepsilon, \theta^\varepsilon)$ such that $U^\varepsilon \rightharpoonup u$, we have $\liminf(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon)) \geq \mathscr{E}(u)$.
(ii) For any u such that $\mathscr{E}(u) < +\infty$, there exists a sequence $(U^\varepsilon, \theta^\varepsilon)$ such that $U^\varepsilon \rightharpoonup u$ and $\limsup(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon)) \leq \mathscr{E}(u)$.

4.4 Proof of the homogenization result

We first prove assertion (i) of Theorem 2.

Proof. We consider a sequence $(U^\varepsilon, \theta^\varepsilon)$ with bounded energy : $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon) \leq M$ (otherwise the result is trivial). Therefore $\varepsilon^2 E_\varepsilon(U^\varepsilon)$ tends to zero. We know from (28) and (29) and Remark 1 that $v^\varepsilon \rightharpoonup\!\!\rightharpoonup v$ and $\chi^\varepsilon_p \rightharpoonup \eta_u$. From Remark 2 we get

$$0 = \liminf_\varepsilon \left(\varepsilon^2 E_\varepsilon(U^\varepsilon)\right) = \liminf_\varepsilon \left(\varepsilon^2 \bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon)\right) \geq \bar{E}(v, \eta_u).$$

Hence $\bar{E}(v,\eta_u) = 0$. Rewriting now $E_\varepsilon(U^\varepsilon)$ as $\sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}(\omega^\varepsilon_{I,p,s,s'})^2$, the energy reads

$$\begin{aligned}\bar{E}_\varepsilon(v^\varepsilon,\chi^\varepsilon) + \bar{F}_\varepsilon(v^\varepsilon,\chi^\varepsilon,\theta^\varepsilon) = \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2} \Bigg(\big(\omega^\varepsilon_{I,p,s,s'}\big)^2 + \frac{\beta^2}{3}\Big(3\big(\theta^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I,s} \\ - \frac{2}{\ell_{p,s,s'}}(v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s} + \chi^\varepsilon_{I,p}) \cdot \tau^\perp_{p,s,s'}\big)^2 + \big(\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s}\big)^2\Big)\Bigg)\end{aligned} \tag{37}$$

Using again (28), (29) and Remarks 1 and 2 we get

$$\begin{aligned}\liminf_\varepsilon \left(\bar{E}_\varepsilon(v^\varepsilon,\chi^\varepsilon) + \bar{F}_\varepsilon(v^\varepsilon,\chi^\varepsilon,\theta^\varepsilon)\right) \geq \int_\Omega \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2} \Bigg(\big(\omega_{p,s,s'}(x)\big)^2 \\ + \frac{\beta^2}{3}\Big(3\big(\theta_{s'}(x) + \theta_s(x) - \frac{2}{\ell_{p,s,s'}}(v_{s'}(x) - v_s(x) + \chi_p(x)) \cdot \tau^\perp_{p,s,s'}\big)^2 \\ + \big(\theta_{s'}(x) - \theta_s(x)\big)^2\Big)\Bigg)\, dx \\ \geq \bar{F}(v,\eta_u,\theta) + \int_\Omega \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}\big(\omega_{p,s,s'}(x)\big)^2 \, dx.\end{aligned} \tag{38}$$

It remains to characterize the limit $\omega_{p,s,s'}$. To that aim, let us introduce the set $\mathcal{D}_A$ of families of distributions in $H^{-1}(\mathbb{R}^2)$:

$$\mathcal{D}_A := \left\{\psi_{p,s,s'} = (w'_s - w_s + \nabla\lambda \cdot p) \cdot \tau_{p,s,s'} : \ (p,s,s') \in \mathcal{A},\ w_s \in \mathrm{L}^2,\ \lambda \in \mathrm{L}^2\right\}$$

and ${\mathcal{D}_A}^{\perp}$ its orthogonal, that is the set of families $(\phi_{p,s,s'})_{(p,s,s')\in\mathcal{A}}$ of functions in $H^1(\mathbb{R}^2)$ such that

$$\sum_{(p,s,s')\in\mathcal{A}} \langle \psi_{p,s,s'}, \phi_{p,s,s'}\rangle = 0$$

for all $\psi_{p,s,s'} \in \mathcal{D}_A$. Let us remark that, for any $\phi \in {\mathcal{D}_A}^{\perp}$ we have

$$\sum_{(p,s,s')\in\mathcal{A}} (\nabla\phi_{p,s,s'} \cdot \mathbf{p})\, \tau_{p,s,s'} = 0. \tag{39}$$

and for any $(w_s) \in \mathrm{L}^2(\mathbb{R}^2, \mathbb{R}^2)^K$,

$$\sum_{(p,s,s')\in\mathcal{A}} ((w_{s'} - w_s) \cdot \tau_{p,s,s'})\, \phi_{p,s,s'} = 0. \tag{40}$$

If we extend ϕ by setting $\phi_{p,s,s'} = 0$ whenever $(p,s,s') \notin \mathcal{A}$ we can rewrite this last equation as

$$\sum_{(p,s,s')\in\mathcal{P}} \tau_{p,s,s'}\, \phi_{p,s,s'} - \tau_{p,s',s}\, \phi_{p,s',s} = 0.$$

Thus, for such functions we have using (40)

$$\begin{aligned}
&\int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \omega_{p,s,s'}(x)\phi_{p,s,s'}(x)\\
&\quad= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \cdot (\phi_{p,s,s'}(\varepsilon y_I)\tau_{p,s,s'})\\
&\quad= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s'}) + \varepsilon^{-1}(v^\varepsilon_{I,s'} - v^\varepsilon_{I,s})\\
&\qquad\qquad\qquad\qquad\qquad + \varepsilon^{-2}(m^\varepsilon_{I+p} - m^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\varepsilon y_I)\tau_{p,s,s'})\\
&\quad= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s'}) + \varepsilon^{-2}(m^\varepsilon_{I+p} - m^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\varepsilon y_I)\tau_{p,s,s'}).
\end{aligned}$$

Considering only smooth functions $\phi_{p,s,s'}$ we can estimate the first addend by

$$\begin{aligned}
&\lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-1}(v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s'}) \cdot (\phi_{p,s,s'}(\varepsilon y_I)\tau_{p,s,s'})\\
&\qquad= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} v^\varepsilon_{I,s'} \cdot (\varepsilon^{-1}(\phi_{p,s,s'}(\varepsilon y_{I-p}) - \phi_{p,s,s'}(\varepsilon y_I))\tau_{p,s,s'})\\
&\qquad= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} v^\varepsilon_{I,s'} \cdot ((-\nabla\phi_{p,s,s'}(\varepsilon y_I) \cdot \mathbf{p})\tau_{p,s,s'}) + O(\varepsilon)\\
&\qquad= -\int_\Omega \sum_{(p,s,s')\in\mathcal{A}} v_{s'}(x) \cdot ((\nabla\phi_{p,s,s'}(x) \cdot \mathbf{p})\tau_{p,s,s'})\, dx\\
&\qquad= \sum_{(p,s,s')\in\mathcal{A}} \langle (\nabla_x v_{s'}(x) \cdot \mathbf{p}) \cdot (\phi_{p,s,s'}(x)\tau_{p,s,s'})\rangle\,.
\end{aligned}$$

The second addend becomes using (39)

$$
\begin{aligned}
&\lim \sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(m^\varepsilon_{I+p} - m^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\varepsilon y_I)\tau_{p,s,s'}) \\
&= \lim \sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2} m^\varepsilon_I \cdot ((\phi_{p,s,s'}(\varepsilon y_{I-p}) - \phi_{p,s,s'}(\varepsilon y_I))\tau_{p,s,s'}) \\
&= \lim \sum_I \sum_{(p,s,s')\in\mathcal{A}} m^\varepsilon_I \cdot ((-\varepsilon^{-1}\nabla\phi_{p,s,s'}(\varepsilon y_I)\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\varepsilon y_I)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'}) \\
&= \lim \sum_I \sum_{(p,s,s')\in\mathcal{A}} m^\varepsilon_I \cdot ((\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\varepsilon y_I)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'}) \\
&= \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} u(x)\cdot((\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(x)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'})\,dx \\
&= \sum_{(p,s,s')\in\mathcal{A}} \frac{1}{2}\left\langle(\nabla\nabla u(x)\cdot\mathbf{p}\cdot\mathbf{p})\cdot(\phi_{p,s,s'}(x)\tau_{p,s,s'})\right\rangle.
\end{aligned}
$$

Collecting these results we obtain that the distribution

$$
\omega^p_{s,s'} - \big(\nabla v_{k'}\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p}\big)\cdot\tau_{p,s,s'}
$$

is orthogonal to all smooth functions in ${\mathcal{D}_A}^\perp$ with compact support in Ω. As they are dense in ${\mathcal{D}_A}^\perp$, we know that there exist some fields w_s and λ in $\mathrm{L}^2(\mathbb{R}^2)$ such that, for any $(p,s,s')\in\mathcal{A}$,

$$
\begin{aligned}
\omega^p_{s,s'} &= \big(w_{s'} - w_s + \nabla(v_{s'}+\lambda)\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p}\big)\cdot\tau_{p,s,s'} \\
&= \big(w_{s'} - w_s + \xi_{u,v+\lambda}\big)\cdot\tau_{p,s,s'}
\end{aligned}
$$

Hence inequality (38) becomes

$$
\liminf_\varepsilon \bar{E}_\varepsilon(v^\varepsilon,\chi^\varepsilon) + \bar{F}_\varepsilon(v^\varepsilon,\chi^\varepsilon,\theta^\varepsilon) \geq \bar{F}(v,\eta_u,\theta) + \bar{E}(w,\xi_{u,v+\lambda}). \tag{41}
$$

Noticing that $\bar{F}(v+\lambda,\eta,\theta) = \bar{F}(v,\eta_u,\theta)$ and $\bar{E}(v+\lambda,\eta) = \bar{E}(v,\eta)$, we get the desired bound. □

We now prove assertion (ii) of Theorem 2.

Proof. Let us now consider a function u such that $\mathcal{E}(u) < +\infty$. By density, we can assume that $u \in C^\infty(\Omega)$. We introduce (v,w,θ) such that $\mathcal{E}(u) = \bar{E}(w,\xi_{u,v}) + \bar{F}(v,\eta_u,\theta)$ and $\bar{E}(v,\eta_u) = 0$. Their existence is ensured by the coercivity and lower semicontinuity of these functionals. The fields (v,w,θ) also belong to $C^\infty(\Omega)$.

Note that $\bar{E}(v,\eta_u) = 0$ implies that, for any $(p,s,s')\in\mathcal{A}$, we have

$$
(v_{s'} - v_s + \nabla u\cdot\mathbf{p})\cdot\tau_{p,s,s'} = 0 \tag{42}
$$

from which we can deduce that

$$
(\nabla v_{s'}\cdot\mathbf{p} - \nabla v_s\cdot\mathbf{p} + \nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p})\cdot\tau_{p,s,s'} = 0. \tag{43}
$$

We now define U^ε and θ^ε by setting

$$
U^\varepsilon_{I,s} := u(\varepsilon y_I) + \varepsilon v_s(\varepsilon y_I) + \varepsilon^2 w_s(\varepsilon y_I) \quad\text{and}\quad \theta^\varepsilon_{I,s} := \theta_s(\varepsilon y_I).
$$

It is clear that $U^\varepsilon \rightharpoonup u$ and $\theta^\varepsilon \rightharpoonup\!\!\rightharpoonup \theta$. Let us compute $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)$. We have, using (42) and (43),

$$\begin{aligned}
\omega^\varepsilon_{I,p,s,s'} &= \varepsilon^{-2}\tau_{p,s,s'} \cdot (U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \\
&= \tau_{p,s,s'} \cdot \Big(\varepsilon^{-2}(u(\varepsilon y_{I+p}) - u(\varepsilon y_I)) + w_{s'}(\varepsilon y_{I+p}) - w_s(\varepsilon y_I) \\
&\qquad + \varepsilon^{-1}(v_{s'}(\varepsilon y_{I+p}) - v_s(\varepsilon y_{I+p})) + \varepsilon^{-1}(v_s(\varepsilon y_{I+p}) - v_s(\varepsilon y_I))\Big) \\
&= \tau_{p,s,s'} \cdot \Big(\varepsilon^{-1}\nabla u(\varepsilon y_I)\cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u(\varepsilon y_I)\cdot\mathbf{p}\cdot\mathbf{p} + w_{s'}(\varepsilon y_{I+p}) - w_s(\varepsilon y_I) \\
&\qquad - \varepsilon^{-1}\nabla u(\varepsilon y_{I+p})\cdot\mathbf{p} + \nabla v_s(\varepsilon y_I)\cdot\mathbf{p}\Big) + O(\varepsilon) \\
&= \tau_{p,s,s'} \cdot \Big(-\frac{1}{2}\nabla\nabla u(\varepsilon y_I)\cdot\mathbf{p}\cdot\mathbf{p} + \nabla v_s(\varepsilon y_I)\cdot\mathbf{p} + w_{s'}(\varepsilon y_{I+p}) - w_s(\varepsilon y_I)\Big) + O(\varepsilon) \\
&= \tau_{p,s,s'} \cdot \Big(\frac{1}{2}\nabla\nabla u(\varepsilon y_I)\cdot\mathbf{p}\cdot\mathbf{p} + \nabla v_{s'}(\varepsilon y_I)\cdot\mathbf{p} + w_{s'}(\varepsilon y_{I+p}) - w_s(\varepsilon y_I)\Big) + O(\varepsilon).
\end{aligned}$$

Hence

$$\omega^\varepsilon_{I,p,s,s'} = \tau_{p,s,s'} \cdot \Big(w_{s'}(\varepsilon y_{I+p}) - w_s(\varepsilon y_I) + (\xi_{u,v})_{p,s'}(\varepsilon y_I)\Big) + O(\varepsilon)$$

and

$$\begin{aligned}
\lim E_\varepsilon(U^\varepsilon) &= \lim \sum_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a^p_{s,s'}}{2}(\omega^\varepsilon_{I,p,s,s'})^2 \\
&= \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \frac{a^p_{s,s'}}{2}\Big((w_{s'}(x) - w_s(x) + (\xi_{u,v})_{p,s,s'})\cdot\tau_{p,s,s'}\Big)^2 = \bar{E}(w,\xi_{u,v}).
\end{aligned} \tag{44}$$

On the other hand

$$\begin{aligned}
&\varepsilon^{-1}\tau^\perp_{p,s,s'}\cdot(U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \\
&\qquad = \tau^\perp_{p,s,s'} \cdot \Big(\varepsilon^{-1}(u(\varepsilon y_{I+p}) - u(\varepsilon y_I)) + v_{s'}(\varepsilon y_{I+p}) - v_s(\varepsilon y_I)\Big) + O(\varepsilon) \\
&\qquad = \tau^\perp_{p,s,s'} \cdot \Big(\nabla u(\varepsilon y_I)\cdot\mathbf{p} + v_{s'}(\varepsilon y_{I+p}) - v_s(\varepsilon y_I)\Big) + O(\varepsilon).
\end{aligned}$$

As $v_s(\varepsilon y_{I+p}) = v_s(\varepsilon y_I) + O(\varepsilon)$ and $\theta_s(\varepsilon y_{I+p}) = \theta_s(\varepsilon y_I) + O(\varepsilon)$, we have

$$\begin{aligned}
\lim F_\varepsilon(U^\varepsilon,\theta^\varepsilon) &= \lim \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}\beta^2}{6}\Big(3\big(\theta_{s'}(\varepsilon y_I) + \theta_s(\varepsilon y_I) \\
&\quad - \frac{2}{\ell_{p,s,s'}}(v_{s'}(\varepsilon y_I) - v_s(\varepsilon y_I) + \nabla u(\varepsilon y_I)\cdot\mathbf{p})\cdot\tau^\perp_{p,s,s'}\big)^2 + \big(\theta_{s'}(\varepsilon y_I) - \theta_s(\varepsilon y_I)\big)^2\Big) \\
&= \bar{F}(v,\eta_u,\theta).
\end{aligned} \tag{45}$$

The result is obtained by collecting (44) and (45). ∎

5 Conclusion

In the limit energy we have identified, namely

$$\mathscr{E}(u) := \inf_{w,v,\theta}\left\{\bar{E}(w,\xi_{u,v}) + \bar{F}(v,\eta_u,\theta);\ \bar{E}(v,\eta_u) = 0\right\}.$$

one has to compute the minimum with respect to three extra kinematic variables. These minima can essentially be computed locally, through "a cell problem". This is clearly the case for θ and w for which solutions depend linearly respectively on $\xi_{u,v}$ and $((v,\eta_u)$. The quadratic constraint $E(v,\eta_u) = 0$ is also easily solved and v takes the form $v = L\cdot\eta_u + \lambda$ with L a linear operator and λ any field in the kernel of this energy. Collecting these results, $\mathscr{E}$ becomes the integral of a quadratic form of the quantities ∇u, $\nabla\nabla u$, λ and $\nabla\lambda$. A priori, the infimum with respect to λ cannot be computed locally and the limit model involves this extra kinematic variable : it is both a generalized continuum model [19] and a strain gradient model. However in many cases, the variable λ does not appear directly in the limit energy and $\nabla\lambda$ can be computed locally. In such cases, the limit model is a pure strain gradient model. The reader must be aware that, depending on the geometry of the considered graph, second gradient effects may be present or not. They are present for instance when considering the graphs represented in figure 3b or in figure 4 but absent when considering the graph of figure 3a. The precise description of the algorithm which makes explicit the limit energy and its application to a complete set of examples is the subject of [1] where the extension of our homogenization theorem 2 to different dimensions of space and periodicity is also provided. In particular we emphasize the fact that the homogenized behaviour of the graph represented in figure 4 corresponds to a complete second gradient material while the graph of figure 3 gives only a couple stress model.

The fact that bending stiffness is by itself a second gradient effect may be misleading. The reader may infer that the presence in our structures of slender slabs, in which Euler-Bernoulli-Navier motions take place, is the source of the homogenized second gradient effects. That is not the case : even when $\beta = 0$, that is when bending stiffness is neglected, second gradient effects remain. They are due to the extensional stiffness of the slender slabs and to a particular design of the periodic cell while the bending stiffness is, on the contrary, the source of the first gradient effects in the homogenized energy and ensures the relative compactness of the considered energies. To understand the nature of the appearance of second gradient through the homogenization process and to conceive new structures with such effects, we recommend the reader to forget bending stiffness and focus on the case $\beta = 0$ reminding that relative compactness can also be ensured by suitable boundary conditions.

6 Appendix

Here we collect the technical proofs of the lemma needed for the reduction of the 2D elastic problem to the discrete one.

Proof of lowerbound Lemma 2: By adding if needed a rigid motion to u, we can restrict our attention to the case $U_2^- = U_2^+ = 0$ and $U_1^- = -U_1^+$. We also remark that, for any $\sigma \in \mathrm{L}^2(\omega)$,

$$\int_\omega \left(\tilde{\mu}\|e(u)\|^2 + \frac{\tilde{\lambda}}{2}\mathrm{tr}(e(u))^2\right) \geq -\int_\omega \left(\frac{1}{4\tilde{\mu}}\|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda}+\tilde{\mu})}\mathrm{tr}(\sigma)^2\right) + \int_\omega \sigma : e(u)$$

Let us choose

$$\sigma = \begin{pmatrix} a + 2b(x_1+c)x_2 & \frac{b}{\sqrt{1+\nu}}(e^2 - x_2^2) \\ \frac{b}{\sqrt{1+\nu}}(e^2 - x_2^2) & 0 \end{pmatrix}$$

with $a = \frac{2Y}{\ell}U_1^+$, $b = -\frac{3Y}{\ell^2}(\theta^- + \theta^+)$ and $c = \frac{\ell}{6}\frac{\theta^+ - \theta^-}{\theta^- + \theta^+}$. Setting $\tilde{Y}(x) := \frac{4\tilde{\mu}(\tilde{\mu}+\tilde{\lambda})}{2\tilde{\mu}+\tilde{\lambda}}$ (which takes the values Y and kY), we have

$$\frac{1}{4\tilde{\mu}}\|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda}+\tilde{\mu})}\mathrm{tr}(\sigma)^2 = \frac{1}{2\tilde{Y}}\Big((a + 2b(x_1+c)x_2)^2 + 2b^2(e^2 - x_2^2)^2\Big).$$

Integrating over the thickness we get

$$\int_\omega \frac{1}{4\tilde{\mu}}\|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda}+\tilde{\mu})}\mathrm{tr}(\sigma)^2 = \int_{-\ell/2}^{\ell/2} \frac{1}{2\tilde{Y}}(2ea^2 + \frac{8e^3}{3}b^2(x_1+c)^2 + \frac{32e^5}{15}b^2)\, dx_1.$$

Direct computations give

$$\int_{-\ell/2}^{\ell/2} \frac{1}{\tilde{Y}(x_1)}\, dx_1 \leq \frac{\ell}{Y}(1 + 2\frac{k'}{k}\frac{e}{\ell})$$

and

$$\int_{-\ell/2}^{\ell/2} \frac{(x_1+c)^2}{\tilde{Y}(x_1)}\, dx_1 \leq \frac{\ell^3}{36Y}\left(3 + \left(\frac{6c}{\ell}\right)^2\right)(1 + 12\frac{k'}{k}\frac{e}{\ell}).$$

Hence

$$\begin{aligned}\int_\omega \frac{1}{4\tilde{\mu}}\|\sigma\|^2 &- \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda}+\tilde{\mu})}\mathrm{tr}(\sigma)^2 \\ &\leq (1 + 82\frac{k'}{k}\frac{e}{\ell})\frac{Ye}{\ell}\Big[(2U_1^+)^2 + \frac{e^2}{3}\left(3(\theta^+ + \theta^-)^2 + (\theta^+ - \theta^-)^2\right)\Big]\end{aligned}$$

On the other hand

$$\int_\omega \sigma : e(u) = \int_{\partial\omega} (\sigma\cdot n)\cdot u$$
$$\geq (1-\frac{e}{2\ell(1+\nu)})\frac{2Ye}{\ell}\Big[(2U_1^+)^2+\frac{e^2}{3}\Big(3(\theta^++\theta^-)^2+(\theta^+-\theta^-)^2\Big)$$
$$-\frac{e}{2\ell(1+\nu)}(v^+-v^-)^2\Big]$$

The lemma is proven by collecting these two results.

□

Proof of upperbound lemma 3: By adding if needed a rigid motion to u, we can restrict our attention to the case $U_2^-+\theta^-k'e = U_2^+-\theta^+k'e = 0$ and $U_1^- = -U_1^+$. In that case we simply have to state for the energy the upperbound

$$\int_\omega \left(\tilde{\mu}\|e(u)\|^2+\frac{\tilde{\lambda}}{2}\mathrm{tr}(e(u))^2\right) \leq \frac{Ye}{\ell}\big(1+C\frac{e}{\ell}\big)\Big[(U_1^+-U_1^-)^2+$$
$$\frac{e^2}{3}\Big(3\gamma^2(\theta^++\theta^-)^2+(\theta^+-\theta^-)^2\Big)\Big] \tag{46}$$

where $\gamma := \frac{\ell-2k'e}{\ell} = 1-2k'\frac{e}{\ell}$. We introduce the continuous piecewise affine function φ defined by $\varphi(x)=1$ if $|x|<\frac{\ell}{2}-2k'e$, $\varphi(x)=0$ if $|x|>\frac{\ell}{2}-k'e$. Then we define u by setting $u(x_1,x_2)=U^-+\theta^-(-x_2,x_1+\frac{\ell}{2})$ if $x_1<-\frac{\ell}{2}+k'e$, $u(x_1,x_2)=U^++\theta^+(-x_2,x_1-\frac{\ell}{2})$ if $x_1>\frac{\ell}{2}-k'e$ and, for $|x_1|<\frac{\ell}{2}-k'e$,

$$u_1(x_1,x_2) := (U_1^+-U_1^-)\frac{x_1}{\gamma\ell}$$
$$-\frac{1}{4\ell^2}\Big(\frac{12x_1^2}{\gamma^2}(\theta^++\theta^-)+\frac{4\ell x_1}{\gamma}(\theta^+-\theta^-)-\ell^2(\theta^++\theta^-)\Big)x_2$$
$$u_2(x_1,x_2) := \frac{\gamma}{8\ell^2}\Big(\frac{2x_1}{\gamma}(\theta^++\theta^-)+\ell(\theta^+-\theta^-)\Big)\Big(\frac{4x_1^2}{\gamma^2}-\ell^2\Big)$$
$$-\frac{\gamma\nu\varphi(x_1)}{\ell^2}\Big(\ell(U_1^+-U_1^-)x_2-\big(\frac{6x_1}{\gamma}(\theta^++\theta^-)+\ell(\theta^+-\theta^-)\big)\frac{x_2^2}{2}\Big)$$

It is straightforward to check that u belongs to $\mathrm{H}^1(\omega,\mathbb{R}^2)$ and some cumbersome but direct computations lead to estimation (46).

□

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