HOMOGENIZATION OF PERIODIC GRAPH-BASED ELASTIC STRUCTURES

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Abstract

In the framework of Γ-convergence and periodic homogenization of highly contrasted materials, we study cylindrical structures made of one material and voids. Interest in high contrast homogenization is growing rapidly but assumptions are generally made in order to remain in the framework of classical elasticity. On the contrary, we obtain homogenized energies taking into account second gradient (i.e. strain gradient) effects. We first show that we can reduce the study of the considered structures to discrete systems corresponding to frame lattices. Our study of such lattices differs from the literature in the fact that we take into account the different orders of magnitude of the extensional and flexural stiffnesses. This allows us to consider structures which would have been floppy when considering only extensional stiffness and completely rigid when considering flexural stiffnesses of the same order of magnitude than the extensional ones. To our knowledge, this paper provides the first rigorous homogenization result in continuum mechanics with a complete second gradient limit energy.

1 Introduction

In [22] it has been proved that highly contrasted heterogeneous elastic materials may lead, through an homogenization process, to materials with very new properties. In particular the order of differentiation of the equilibrium equations may be much higher for the homogenized material than they were for the heterogeneous one. However

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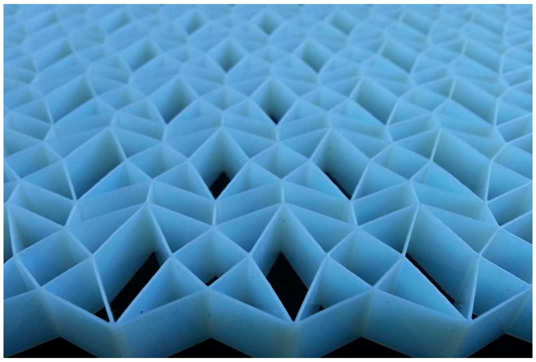

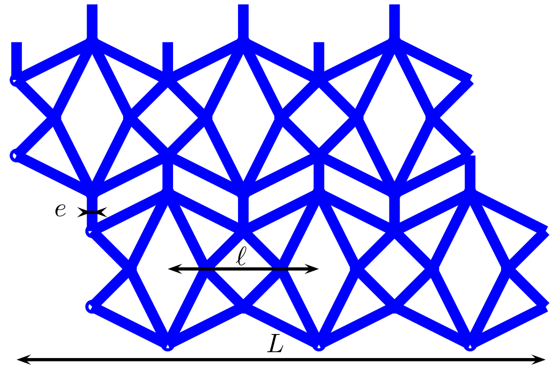

Figure 1: A cylindrical 3D elastic structure based on a thickened periodic planar graph. The basis of the cylinder is a thickened periodic graph Ω^ε with a global size L, a period size ℓ, a thickness of edges e which are three lengths with different orders of magnitude.

very few explicit examples have been given in which such a phenomenon appears. In [44], [11], [19] the homogenized material becomes a second gradient one : the elastic energy depends on the second gradient of the displacement instead of the first one only. However all these results fall under the framework of couple stress theory, [51], [52], [38], [39] : the dependence with respect to the second gradient of the displacement is limited to dependence on the gradient of the skew part of the gradient of the displacement only. To our knowledge complete second gradient media have been obtained, up to now, only through homogenization of discrete systems based on pantographic structures [6], [7], [49].

Second gradient materials are, among other generalized continuum models widely used [28], [27], [36], [29], ... Their very rich behavior allows for instance to regularize and thus to study precisely the parts of materials where the deformation tends to concentrate, [53], [30] (inter-phases, [40], [20], [32], [48], porous media, [47], fractures, [2], [3], damage and plasticity, [54], [45]). However the second gradient properties are scarcely measured directly, [8], [9], [33] nor rigorously interpreted from a microscopic point of view. Mechanicians have no tool for conceiving second gradient materials with chosen properties.

The aim of this paper is to provide such a tool. It is is not question here to solve all highly contrasted periodic homogenization problems but to describe a set of situations sufficiently large for making clear how appear second gradient effects through the homogenization process.

It is important to remark that second gradient properties are generally obtained in the literature as *corrections* to the homogenized model : they do not appear in the limit energy but as a next term in an asymptotic development [15], [5] [50] with respect to the size of the heterogeneities. There is an essential difference of nature between second gradient limit energies and second gradient terms in an asymptotic development : in the first case the model contains a finite intrinsic length while in the second case the intrinsic lengths are infinitely small. Another fact enlightens the difference between both approaches : second gradient asymptotic developments can be obtained even when homogenizing conduction problems [5] while it has been proved in [21] and [22] that second gradient limit energies were possible for elasticity homogenization problems but unattainable when considering conduction problems. Asymptotic developments are difficult to interpret and applying them to real problems leads to many questions. For instance the sign of the second gradient terms in these developments may change and lead to ill-posed equilibrium problems. For instance still, it seems difficult to justify the fact that the maximum principle applies in heterogeneous conduction problems but would not apply when replacing them by their homogenized second gradient development. All these problems cannot arise when applying our results. Our limit energies, owing to standard properties of Γ-convergence, are necessarily positive lower semi-continuous quadratic forms.

We consider structures made of a periodic arrangement of welded thin walls (see for instance figure 1) : they are cylinders (see Figure 1a) the basis of which is a thickened periodic planar graph (see Figure 1b).

We study, in the framework of Γ-convergence, the homogenization of these structures and rigorously determine the second gradient effects. To that aim we make some modeling assumptions which, of course, can be questioned when applied to the real structure of figure 1:

- First we assume that the structure is made of a homogeneous isotropic linear elastic material. We thus implicitly forbid the possibility of any micro buckling effect.

- We also consider that the structure is solicited in the plane of the graph and assume that we are in the conditions of plane strain elasticity. This assumption, valid only when the height of the structure is large enough, allows us to reduce the problem to a bi-dimensional one : a linear elastic problem set in a thickened periodic planar graph; more precisely, in the intersection of this thickened graph and a bounded domain $\Omega \subset \mathbb{R}^2$.

- As our goal is to determine the effective properties of the material, we have to suppose that the size ℓ of the period of the graph is small compared to the characteristic size L of the domain Ω. This is the standard asymptotic homogenization assumption.
$$\epsilon = \ell/L << 1$$

- We consider that the thickness e of the walls the structure is made of (i.e. the thickness of the graph) is small compared with ℓ. Hence the 2D elastic problem we consider contains two small dimensionless parameters which we let tend to zero :
$$\delta = e/\ell << 1$$
This assumption is essential : otherwise the standard homogenization results would be valid and the effective properties of the material would be those of a classical (may be non isotropic) elastic material. This assumption will also have a practical effect on our mathematical arguments : as it implies that the edges are slender rectangles, we can, using the theory of slender elastic structures, reduce our problem to the study of a discrete system.

- The two limits $\epsilon \to 0$ and $\delta \to 0$ do not commute and we have to specify the way they simultaneously go to zero : we assume that
$$\delta = \beta\epsilon$$
with $\beta > 0$ fixed. Indeed, this case is critical : the cases $\delta = \epsilon^\alpha$ with $\alpha > 1$ or $\alpha < 1$ can be deduced from our results by letting in a further step β tend to zero or to infinity.

- Finally we have to specify the order of magnitude of the rigidity of the material our structure is made of. We emphasize that speaking of the order of magnitude of the stiffness of the material takes sense only if we compare it to some force. In other words, making an assumption over the elastic rigidity is equivalent to making an assumption over the order of magnitude of the applied external forces. As the total volume of our structure tends to zero with δ, it is clear that we need a strong rigidity of the material if we desire to resist to forces of order one. Different assumptions can be made which correspond to different experiments. This is not surprising : the reader accustomed for instance to the 3D-2D or 3D-1D reduction of models for plates or beams, knows that changing the assumptions upon the order of magnitude of the elasticity stiffness of the material changes drastically the limit model. If the structure cannot resist to some applied forces (like a membrane cannot resist to transverse forces), it may resist to them after a suitable scaling of the material properties (like the membrane model is replaced by the Kirchhoff-Love plate model). Simultaneously some mobility may disappear (like the Kirchhoff-Love plate becomes inextensible).
In this paper we are interested in the case where the Lamé coefficients (μ, λ) of the material tend to infinity like $\delta^{-1}\epsilon^{-2}$:
$$\mu = \frac{\mu_0}{\beta\epsilon^3}, \qquad \lambda = \frac{\lambda_0}{\beta\epsilon^3} \tag{1}$$

- For sake of simplicity we assume free boundary conditions along the whole boundary of Ω. The discussion about the different boundary conditions which can be assumed and the way they pass to the limit would make this paper too long. As usual when dealing with Neumann-type boundary conditions, we have to assume that the external forces applied to the structure are balanced and we ensure uniqueness of the equilibrium solution by imposing zero mean rigid motion.

The paper is organized as follows. In Section 2, we first describe precisely the geometry we are interested in by introducing in a sparse way the graphs on which our 3D structures are based. Assuming a plane strain state we state the elastic problem in a 2D domain which corresponds to a thickened graph.

Several studies deal with this problem (see for instance [13], [14], [23] [43]) but the assumptions which are made in these papers are stronger than ours (either limiting the energy to conduction problems or fixing a thickness for the walls of the same order of magnitude as the size of the periodic cell or limiting the stiffness of material the structure is made of to a too small order of magnitude) and their results are thus limited to classical (first gradient) homogenized energy.

In Section 3, we prove that our 2D elastic problem has the same Γ-limit as an equivalent discrete problem set on the nodes of the graph. Both extensional and flexural stiffnesses must be taken into account even if the flexural rigidity is much lower than the extensional one. This part is rather technical and the sketches of the proofs (which are more or less standard) are postponed to the Appendix.

In Section 4 we attack the problem of finding the Γ-limit of the discrete energy. We study the problem from the variational point of view adapting to our case the tools of Γ-convergence, [25], [17] and double-scale limit, [41], [4] which have shown their efficiency for treating many different problems of homogenization. The topology we use is rather weak but it is sufficient to ensure at least, that the equilibrium of the structure under the action of forces applied at the nodes of the structure will be well described by the equilibrium of the limit model. This discrete homogenization problem has been studied in [34], [35], [46] and in different contexts in [37], [18]. Again only first gradient limit models have been obtained. The point is that, in all these papers, the order of magnitude of the different types of interaction are supposed not to interfere with the homogenization asymptotic process (see Remark 7.5 of Ref. [34], Remark (2.7) of Ref. [37] or Ref. [18]) while here the ratio between flexural and extensional interactions is comparable to the homogenizing small parameter.

In this paper we do not exhaust all interesting questions about our structures : it has been shown in [49] that the types of actions (external forces, external distributions, boundary distributions,. . .) which can be applied to second gradient materials were much richer than the boundary conditions and external forces considered here. However the case we study is sufficient to enlighten the way second gradient effects can arise through the homogenization procedure. In section 5 we give an example where the limit energy is a complete second gradient one. By "complete" we mean that it does not reduce to a couple-stress model where the energy involves only the gradient of the skew part of the gradient of the displacement. To our knowledge, this is the first rigorous homogenization result with a complete second gradient limit energy.

2 Initial problem, description of the geometry

2.1 The graph

The geometry we consider is based on a periodic planar graph. We adopt a description close to the one used in [31]. Such a graph is determined by

- a prototype cell Y containing a finite number K of nodes the position of which is denoted y_s, $s \in \{1, \dots, K\}$;
- two independent periodicity vectors t_1, t_2. As the graph will be re-scaled, we can assume without loss of generality that $t_1 \times t_2 = 1$ (i.e. the area $|\mathrm{Y}| = 1$). Introducing, for $I = (i, j) \in \mathbb{N}^2$, the points $y^\varepsilon_{I,s} := \varepsilon(y_s + it_1 + jt_2)$, the set of nodes of the graph is
$$\left\{y^\varepsilon_{I,s} \ : \ I \in \mathbb{N}^2, \ s \in \{1, \dots, K\}\right\}$$
We use $y^\varepsilon_I := \frac{1}{K}\sum_{s=1}^{K} y^\varepsilon_{I,s}$ as a reference point for the cell I;
- five $K \times K$ matrices a_p taking value in $\mathbb{R}^+$ defining the edges of the graph : an edge links nodes $y_{I,s}$ and $y_{I+p,s'}$ as soon as $a_{p,s,s'} > 0$. Here p belongs to the set[1]
$$\mathcal{P} := \{(0, 0), (1, 0), (0, 1), (1, 1), (1, -1)\}.$$
We denote $\mathbf{p} := p_1t_1 + p_2t_2 \in \{0, t_1, t_2, t_1 + t_2, t_1 - t_2\}$ the corresponding vector so that $y^\varepsilon_{I+p,s} = y^\varepsilon_{I,s} + \varepsilon\mathbf{p}$. We introduce the set of multi-indices corresponding to all edges :
$$\mathcal{A} := \{(p, s, s') : \ p \in \mathcal{P}, \ 1 \le s \le K, \ 1 \le s' \le K, \ a_{p,s,s'} > 0\}.$$
For any $(p, s, s') \in \mathcal{A}$ we introduce the rescaled length and direction of the edge by setting
$$\ell_{p,s,s'} := \varepsilon^{-1}\|y^\varepsilon_{I+p,s'} - y^\varepsilon_{I,s}\|, \quad \text{and} \quad \tau_{p,s,s'} := \frac{y^\varepsilon_{I+p,s'} - y^\varepsilon_{I,s}}{\varepsilon\ell_{p,s,s'}}.$$

[1] Note that, owing to periodicity, only half of the neighbors of a cell have been considered. It is also important to notice that there is no loss of generality (as soon as we assume that the range of interactions is finite) in assuming that a cell is interacting only with its closest neighbors. Indeed we can always choose a prototype cell large enough for this assumption to become true.

- a bounded convex domain Ω in $\mathbb{R}^2$. We assume that Ω has measure 1 (choice of the unit length) and so that $L = 1$. We denote $\mathcal{I}^\varepsilon$ be the set of cells which lie sufficiently inside the domain :

$$\mathcal{I}^\varepsilon := \Big\{ I;\ y_I^\varepsilon \in \Omega \text{ and } d(y_I^\varepsilon, \partial\Omega) > \sqrt{\varepsilon} \Big\}$$

  (where d stands for the Euclidian distance), $\mathring{G}^\varepsilon$ the set of nodes of these cells and G^ε the union of the edges which link them

$$\mathring{G}^\varepsilon := \bigcup_{I\in\mathcal{I}^\varepsilon} \bigcup_{s=1}^{K} \{y_{I,s}^\varepsilon\}, \qquad G^\varepsilon := \bigcup_{I\in\mathcal{I}^\varepsilon} \bigcup_{(p,s,s')\in\mathcal{A}} [y_{I,s}^\varepsilon, y_{I+p,s'}^\varepsilon].$$

  The number N^ε of such cells is equivalent to ε^{-2} and we will denote in the sequel the mean value of any quantity φ defined on $\mathcal{I}^\varepsilon$ by

$$\frac{}{}\!\!\!\!\sum_I \varphi_I := \frac{1}{N^\varepsilon} \sum_{I\in\mathcal{I}^\varepsilon} \varphi_I \sim \varepsilon^2 \sum_{I\in\mathcal{I}^\varepsilon} \varphi_I$$

  The planar elastic problem will be set in the thickened graph:

$$\Omega^\varepsilon := \left\{ x \in \Omega;\ d(x, G^\varepsilon) < \beta\varepsilon^2 \right\}, \tag{2}$$

  where the thickened nodes $B_{I,s}^\varepsilon$

$$B_{I,s}^\varepsilon := \{x;\ d(x, y_{I,s}^\varepsilon) < \beta\varepsilon^2\}$$

  play an essential role.

Restrictive assumptions : Not all interaction matrices are admissible :

- There is no crossing or overlapping of different edges : for any (p, s, s') and $(\tilde{p}, \tilde{s}, \tilde{s}')$ in $\mathcal{A}$,

$$[y_{I,s}^\varepsilon, y_{I+p,s'}^\varepsilon] \cap [y_{\tilde{I},\tilde{s}}^\varepsilon, y_{\tilde{I}+\tilde{p},\tilde{s}'}^\varepsilon] \not\subset \{y_{I,s}^\varepsilon, y_{I+p,s'}^\varepsilon\} \Rightarrow (\tilde{I}, \tilde{s}, \tilde{p}) = (I, s, p).$$

  This assumption results from the cylindrical shapes we are studying but is not fundamental. One could design multilayered structures, allowing crossing of interactions. The reduction to a discrete problem would then have to be adapted to this case.

- We are not interested by lattices which are made of several disconnected lattices. So we assume that the edges connect all the nodes of the structures. More precisely we assume that, for any $p \in \mathcal{P}$ and any $(s, s') \in \{1, \ldots, K\}^2$, there exist a finite path in the graph which joins the node $y_{I,s}^\varepsilon$ to the node $y_{I+p,s'}^\varepsilon$ that is a finite sequence $(s_1, \ldots, s_{r+1})$ in $\{1, \ldots, K\}$, $(p_1, \ldots, p_r)$ in $\mathcal{P}$, $(\epsilon_1, \ldots, \epsilon_r)$ in $\{-1, 1\}$ such that $s_1 = s$, $s_{r+1} = s'$, $\sum_{i=1}^r \epsilon_i \mathbf{p}_i = \mathbf{p}$,

$$\epsilon_i > 0 \Rightarrow (p_i, s_i, s_{i+1}) \in \mathcal{A} \qquad \text{and} \qquad \epsilon_i < 0 \Rightarrow (p_i, s_{i+1}, s_i) \in \mathcal{A}.$$

  Note that, for nodes $y_{I,s}^\varepsilon$ and $y_{I+p,s'}^\varepsilon$ lying sufficiently inside Ω, the joining path is independent of I but that this path may have to be modified for nodes lying close to the boundary. We assume that such a path can always be chosen in a finite set of paths.

  As we will see in Lemma 4 this assumption implies the strong-2-connectedness in the sense of [14].

  Even if it seems clear when considering Figures 2 and 3, it is not so easy to check if a structure is connected. This has been studied in [10] where algorithms for this checking are provided.

Some examples of non connected graphs are given in figure 2 while examples of admissible graphs are given in figures 1 and 3.

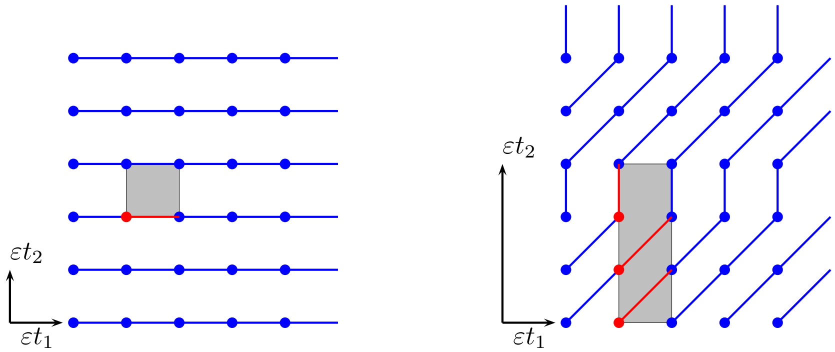

Figure 2: Non connected structures for which relative compactness is not ensured.

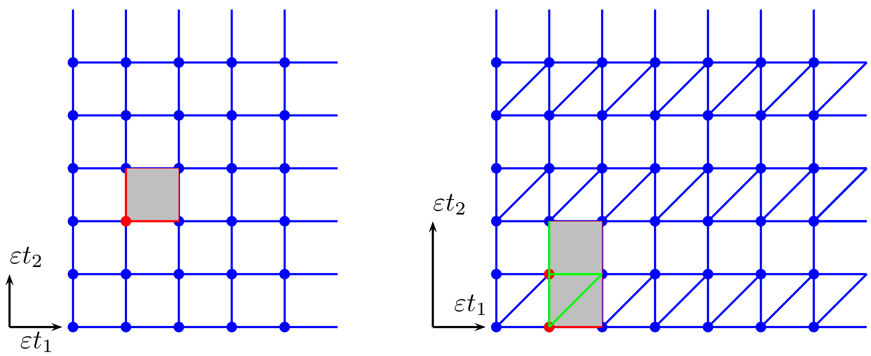

Figure 3: Admissible structures.

2.2 The 2D elastic problem

As we have chosen $L = 1$, our assumptions resume in

$$\ell = \varepsilon, \quad e = \beta\varepsilon^2, \quad \mu = \frac{\mu_0}{\beta\epsilon^3}, \quad \lambda = \frac{\lambda_0}{\beta\epsilon^3}.$$

The elastic energy $\mathscr{E}_\varepsilon$ is defined, for any displacement field $u \in \mathrm{L}^2(\Omega^\varepsilon, \mathbb{R}^2)$ with zero mean rigid motion, by

$$\mathscr{E}_\varepsilon(u) := \begin{cases} \dfrac{1}{\beta\varepsilon^3} \displaystyle\int_{\Omega^\varepsilon} \left(\mu_0 \|e(u)\|^2 + \dfrac{\lambda_0}{2} \mathrm{tr}(e(u))^2 \right) dx & \text{if } u \in \mathrm{H}^1(\Omega^\varepsilon, \mathbb{R}^2), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

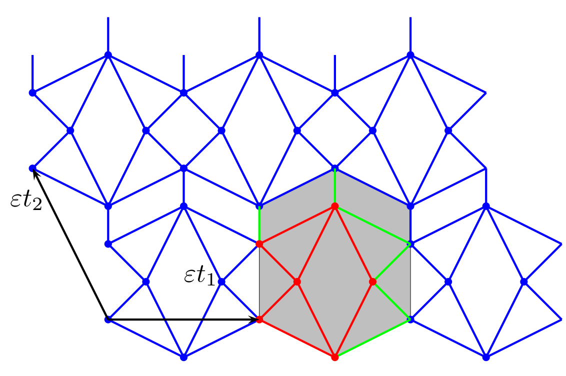

Figure 4: Another admissible structure : the planar graph corresponding to Figure 1.

Here $e(u)$ denotes the symmetric part of the gradient of u ($e(u) = (\nabla u + \nabla^t u)/2$ is the linearized strain tensor), $\mathrm{tr}(e(u))$ denotes the trace of the matrix $e(u)$. To Lamé coefficients (which satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 > 0$), we associate Young modulus

$$Y = \frac{Y_0}{\beta\varepsilon^3} \quad \text{where} \quad Y_0 := \frac{4\mu_0(\mu_0 + \lambda_0)}{2\mu_0 + \lambda_0}$$

and Poisson ratio

$$\nu := \frac{\lambda}{2\mu + \lambda} = \nu_0 := \frac{\lambda_0}{2\mu_0 + \lambda_0}.$$

The reader may have noticed that the values of the positive coefficients $a_{p,s,s'} > 0$ of the interaction matrices were, up to now, irrelevant (as soon as they remain positive). We now fix them by setting

$$a_{p,s,s'} = \frac{2Y_0}{\ell_{p,s,s'}} \tag{4}$$

2.3 Convergence

In order to study the homogenization of the considered structures, we need to specify the way we pass to the limit of a sequence of fields (u^ε) with finite energy $\mathscr{E}_\varepsilon(u^\varepsilon) < +\infty$. Indeed each term is defined on a different domain Ω^ε. To that aim, we first introduce the operator $u \to \bar{u}$, which to any field $u \in \mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$ associates the family $\bar{u}$ of mean values defined for $I \in \mathcal{I}^\varepsilon$ and $s \in \{1, \dots, K\}$ by

$$\bar{u}_{I,s} := \fint_{B^\varepsilon_{I,s}} u(x)\, dx := \frac{1}{|B^\varepsilon_{I,s}|} \int_{B^\varepsilon_{I,s}} u(x)\, dx. \tag{5}$$

Note that this operator which maps $\mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$ onto the set $\mathcal{V}_\varepsilon$ of functions defined on $\mathcal{I}^\varepsilon \times \{1, \dots, K\}$ actually depends on ε, even if the notation does not recall it.

Then we define the convergence of a sequence of families of vectors $(Z^\varepsilon_I)_{I \in \mathcal{I}^\varepsilon}$: We say that (Z^ε) converges to the measurable function z, and we write $Z^\varepsilon \rightharpoonup z$, when the following weak* convergence of measures holds true :

$$\sum_I Z^\varepsilon_I \delta_{y^\varepsilon_I} \overset{*}{\rightharpoonup} z(x)\, dx \tag{6}$$

where δ_y stands for the Dirac measure at point y.

Finally we say that the sequence of functions (u^ε) (where $u^\varepsilon \in \mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$) converges to u when, for all $s \in \{1, \dots, K\}$, $(\bar{u}^\varepsilon)_{I,s} \rightharpoonup u$. As no confusion can arise, we simply write $u^\varepsilon \rightharpoonup u$.

Remark 1. *The convergence (6) means that, for all $\varphi \in C^0(\Omega)$,*

$$\sum_I Z^\varepsilon_I \varphi(y^\varepsilon_I) \to \int_\Omega z(x)\varphi(x)\, dx \tag{7}$$

When applying this notion to sequences (Z^ε) such that $\sum_I \|Z^\varepsilon_I\|^2$ is bounded, we are thus assured (see [24] Lemma 10.1) that a subsequence converges to some $z \in \mathrm{L}^2(\Omega)$. In view of (7), we note that we can replace in (6) the Dirac measure $\delta_{y^\varepsilon_I}$ by $\delta_{y^\varepsilon_{I,s}}$ or even $\delta_{y^\varepsilon_{I+p,s}}$. Indeed $\varphi(y^\varepsilon_{I+p}, s') - \varphi(y^\varepsilon_{I,s}, s) = o(1)$.

Remark 2. *The convergence of measures (6) when holding for any $s \in \{1, \dots, K\}$ is closely related to the double-scale convergence as defined by [41] or [4]. Here our discrete variable s plays the role of the fast variable. In that case, for any convex lower semi continuous function Φ we have*

$$\liminf_\varepsilon \sum_I \frac{1}{K} \sum_{s=1}^K \Phi(Z^\varepsilon_{I,s}) \geq \int_\Omega \Phi(z(x))\, dx. \tag{8}$$

(see [12] Lemma 3.1).

Remark 3. *The choice of this convergence allows for the direct application of our homogenization result to the computation of the equilibrium when external forces are applied at the "nodes" of the structure. More precisely to forces fields f^ε of the type*

$$f_s^\varepsilon(x) = \frac{1}{N^\varepsilon \pi \beta^2 \varepsilon^4} \sum_{I\in\mathcal{I}} \sum_{s=1}^{K} f(y_{I,s}) 1_{B^\varepsilon_{I,s}}(x)$$

where f is a continuous field. Indeed the external potential due to such forces corresponds to a continuous perturbation of the energy with respect to the considered convergence. The reader can refer to [25] or [17] for a description of properties of Γ-convergence.

3 Reduction to a discrete problem

We prove in this section that the considered structure can be studied as a discrete one. To any function (U,θ) defined on the nodes of the graph (U being vector valued while θ is scalar), we associate the energies

$$E_\varepsilon(U) := \sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}} \frac{a^p_{s,s'}}{2} \left(\frac{U_{I+p,s'} - U_{I,s}}{\varepsilon} \cdot \tau_{p,s,s'} \right)^2 \tag{9}$$

$$F_\varepsilon(U,\theta) := \varepsilon^2 \sum_{(I,p,s,s')\in\mathcal{I}^\varepsilon\times\mathcal{A}} \frac{a^p_{s,s'}\beta^2}{6} \Big(3\big(\theta_{I+p,s'} + \theta_{I,s} - \frac{2}{\ell_{p,s,s'}} \frac{U_{I+p,s'} - U_{I,s}}{\varepsilon} \cdot \tau^\perp_{p,s,s'}\big)\big)^2 + (\theta_{I+p,s'} - \theta_{I,s})^2 \Big). \tag{10}$$

The sum $E_\varepsilon + F_\varepsilon$ corresponds to the elastic energy of a system of nodes linked by extensional and flexural bars. This section is devoted to the proof of the following theorem which states that the Γ-limit of the initial sequence of 2D elastic energies is identical to the limit of the sequence of these discrete energies.

Theorem 1. *The sequences $(\mathscr{E}_\varepsilon)$ and $(E_\varepsilon + F_\varepsilon)$ share the same Γ-limit $\mathscr{E}$. Indeed, for any measurable function u, we have*

(i) $$\inf_{u^\varepsilon}\{\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \ : \ u^\varepsilon \rightharpoonup u\} \geq \inf_{U^\varepsilon,\theta^\varepsilon}\{\liminf_{\varepsilon\to 0}(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon));\ U^\varepsilon \rightharpoonup u\}$$

(ii) $$\inf_{u^\varepsilon}\{\limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \ : \ u^\varepsilon \rightharpoonup u\} \leq \inf_{U^\varepsilon,\theta^\varepsilon}\{\limsup_{\varepsilon\to 0}(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon,\theta^\varepsilon)));\ U^\varepsilon \rightharpoonup u\}$$

In order to prove this result, we first recall some results concerning the elastic behavior of a thin rectangle which are well known in an asymptotic form but that we need here to state more precisely in order to be able to apply them to the whole structure. Their proofs are postponed to the Appendix.

3.1 Estimations for an elastic rectangle

We use the orthonormal basis (e_1,e_2) in $\mathbb{R}^2$ and consider the rectangle $\omega := [-\ell/2, \ell/2] \times [-e, +e]$ (with $e < \ell/4$).

To any function $u \in \mathrm{H}^1(\omega)$ we associate

$$U(x_1) := \frac{1}{2e}\int_{-e}^{e} u(x_1,x_2)\,dx_2, \quad \theta(x_1) := -\frac{3}{2e^3}\int_{-e}^{e} u_1(x_1,x_2)\,x_2\,dx_2,$$

$$v(x_1) := \frac{3}{4e^3}\int_{-e}^{e} (u_2(x_1,x_2) - U_2(x_1))(e^2 - x_2^2)\,dx_2.$$

and

$$W := \frac{1}{\pi e^2}\int_{B(0,e)} u(x_1,x_2)\,dx_1dx_2, \quad \phi := \frac{1}{\pi e^2}\int_{B(0,e)} \frac{\partial_1 u_2 - \partial_2 u_1}{2}(x_1,x_2)\,dx_1dx_2.$$

Lemma 1. *There exists a constant C independent of e such that, for any $u \in \mathrm{H}^1(B(0,e),\mathbb{R}^2)$*

$$\|U(0) - W\|^2 \leq C\int_{B(0,e)} \|e(u)\|^2 dx, \quad \|\theta(0) - \phi\|^2 \leq Ce^{-2}\int_{B(0,e)} \|e(u)\|^2 dx,$$

$$\|v(0)\|^2 \leq C\int_{B(0,e)} \|e(u)\|^2 dx.$$

Proof. By rescaling we can reduce to the case $e = 1$. Let us assume by contradiction that there exists a sequence u^n such that $\int_{B(0,e)} \|e(u^n)\|^2 dx$ tends to zero while one of the quantities $\|U^n(0) - W^n\|^2$, $\|v^n(0)\|^2$ and $\|\theta^n(0) - \phi^n\|^2$ do not tend to zero. Adding if needed a rigid motion to u^n, we can assume $W^n = 0$ and $\phi^n = 0$. From Korn and Poincaré-Wirtingen inequalities we know that $\|u^n\|_{\mathrm{H}^1(B(0,e),\mathbb{R}^2)}$ tends to zero. A trace theorem ensures that u^n tends to zero in $\mathrm{H}^{1/2}(\{0\} \times [-e, e], \mathbb{R}^2)$ and thus in $\mathrm{L}^2(\{0\} \times [-e, e], \mathbb{R}^2)$. In consequence, contrarily to what we have assumed, $U^n(0)$, $v^n(0)$ and $\theta^n(0)$ tend to zero. $\square$

Now, let $0 \le k < 1 < k' < \ell/(2e)$. In ω, we consider the piecewise constant functions $(\tilde{\mu}, \tilde{\lambda})$ defined by $\tilde{\mu}(x_1, x_2) = \mu$, $\tilde{\lambda}(x) = \lambda$ if $|x_1| < \ell/2 - k'e$, $\tilde{\mu}(x) = k\mu$, $\tilde{\lambda}(x) = k\lambda$, otherwise and we denote respectively (U^-, θ^-, v^-) and (U^+, θ^+, v^+) the values of (U, θ, v) at $x_1 = -\frac{\ell}{2}$ and $x_1 = +\frac{\ell}{2}$.

Lemma 2. *There exists a constant C depending only on k, k' and ν such that, for any $u \in \mathrm{H}^1(\omega)$,*

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \ge \frac{Ye}{\ell} \big(1 - C\frac{e}{\ell}\big) \times \Big[(U_1^+ - U_1^-)^2 + \frac{e^2}{3} \Big(3(\theta^+ + \theta^- - 2\frac{U_2^+ - U_2^-}{\ell})^2 + (\theta^+ - \theta^-)^2 \Big) - \frac{e}{\ell}(v^+ - v^-)^2 \Big]$$

Lemma 3. *There exists a constant C depending only on k, k' and ν such that, for any U^+, U^- in $\mathbb{R}^2$ and θ^+, θ^- in $\mathbb{R}$ there exists $u \in \mathrm{H}^1(\omega, \mathbb{R}^2)$ satisfying $u(x_1, x_2) = U^- + \theta^-(-x_2, x_1)$ if $x_1 < -\frac{\ell}{2} + k'e$, $u(x_1, x_2) = U^+ + \theta^+(-x_2, x_1)$ if $x_1 > \frac{\ell}{2} - k'e$ and*

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \le \frac{Ye}{\ell} \big(1 + C\frac{e}{\ell}\big) \Big[(U_1^+ - U_1^-)^2 + \frac{e^2}{3} \Big(3(\theta^+ + \theta^- - 2\frac{U_2^+ - U_2^-}{\ell})^2 + (\theta^+ - \theta^-)^2 \Big) \Big]$$

Proofs of these two lemmas are given in the Appendix.

3.2 Estimation for the whole structure

We can now prove Theorem 1.

Proof. We first notice that the number of edges which concur at a node $y_{I,s}$ of the graph is bounded by $9K$. We set $k = (9K)^{-1}$. Therefore there exists a uniform lowerbound $\theta_m > 0$ for the angles between these different edges. The thickened edges of Ω^ε concurring at node $y_{I,s}$ do not intersect out of the disk of center $y_{I,s}$ and radius $k'e$ with $k' = (\sin(\theta_m/2))^{-1}$. We consider on Ω^ε the functions $(\tilde{\mu}, \tilde{\lambda})$ defined by $\tilde{\mu}(x) := \mu_0$, $\tilde{\lambda}(x) := \lambda_0$, if $d(x, \mathring{G}^\varepsilon) > k'\beta\varepsilon^2$, $\tilde{\mu}(x) = k\mu_0$, $\tilde{\lambda}(x) = k\lambda_0$, otherwise.

Let u^ε be any sequence of displacement fields with bounded elastic energy $\mathscr{E}_\varepsilon(u^\varepsilon) \le M$ and converging to some function u. Our choice for k and k' allows us to split the energy :

$$\begin{aligned} \mathscr{E}_\varepsilon(u^\varepsilon) &= \frac{1}{\beta\varepsilon^3} \int_{\Omega^\varepsilon} \left(\mu_0 \|e(u^\varepsilon)\|^2 + \frac{\lambda_0}{2} \mathrm{tr}(e(u^\varepsilon))^2 \right) dx \\ &\ge \frac{1}{\beta\varepsilon^3} \sum_{(I,p,s,s') \in \mathcal{I}^\varepsilon \times \mathcal{A}} \int_{S_{I,p,s,s'}} \left(\tilde{\mu} \|e(u^\varepsilon)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u^\varepsilon))^2 \right) dx. \end{aligned}$$

where $S_{I,p,s,s'}$ denotes the rectangle with mean line $[y_{I,s}, y_{I+p,s'}]$ and thickness $2\beta\varepsilon^2$. Applying Lemma 2 to each term of this sum, we get

$$\begin{aligned} \mathscr{E}_\varepsilon(u^\varepsilon) \ge & \frac{1}{2\varepsilon^2} \big(1 - \frac{C\beta}{\ell_m}\varepsilon\big) \sum_{(I,p,s,s') \in \mathcal{I}^\varepsilon \times \mathcal{A}} a_{p,s,s'} \Big[((U^{\varepsilon+}_{I,p,s,s'} - U^{\varepsilon-}_{I,p,s,s'}) \cdot \tau_{p,s,s'})^2 - \frac{\beta\varepsilon}{2\ell_{p,s,s'}} (v^{\varepsilon+}_{I,p,s,s'} - v^{\varepsilon-}_{I,p,s,s'})^2 \\ & + \frac{\beta^2\varepsilon^2}{3} \Big(3\big(\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} + \theta^{\varepsilon-}_{I,p,s,s'}) - \frac{(U^{\varepsilon+}_{I,p,s,s'} - U^{\varepsilon-}_{I,p,s,s'}) \cdot \tau^\perp_{p,s,s'}}{\ell_{p,s,s'}}\big)^2 + (\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} - \theta^{\varepsilon-}_{I,p,s,s'}))^2 \Big) \Big] \end{aligned}$$

where $U^{\varepsilon+}_{I,p,s,s'}$, $U^{\varepsilon-}_{I,p,s,s'}$, $v^{\varepsilon+}_{I,p,s,s'}$, $v^{\varepsilon-}_{I,p,s,s'}$, $\theta^{\varepsilon+}_{I,p,s,s'}$, $\theta^{\varepsilon-}_{I,p,s,s'}$ are the quantities associated to u^ε on the rectangle $S_{I,p,s,s'}$ as in Lemma 2.

On the other hand, Lemma 1 states that, for any (p, s, s'), the quantities $\sum_I \|U^{\varepsilon-}_{I,p,s,s'} - \bar{u}^\varepsilon_{I,s}\|^2$, $\sum_I \|U^{\varepsilon+}_{I,p,s,s'} - \bar{u}^\varepsilon_{I+p,s}\|^2$, $\sum_I |v^{\varepsilon-}_{I,p,s,s'}|^2$, $\sum_I |v^{\varepsilon+}_{I,p,s,s'}|^2$, $\sum_I |\varepsilon(\theta^{\varepsilon-}_{I,p,s,s'} - \phi^\varepsilon_{I,s})|^2$ and $\sum_I |\varepsilon(\theta^{\varepsilon+}_{I,p,s,s'} - \phi^\varepsilon_{I+p,s})|^2$ are all bounded by

$\sum_I \int_{B^\varepsilon_{I,s}} \|e(u)\|^2$ and thus by $C\varepsilon^3$ with $C = \frac{M\beta}{\min(\mu_0,\mu_0+\lambda_0)}$. Here $\phi^\varepsilon_{I,s}$ is the quantity associated to u^ε on the disk $B^\varepsilon_{I,s}$ as in Lemma 1. Hence

$$\begin{aligned}\mathscr{E}_\varepsilon(u^\varepsilon) \geq \frac{1}{2\varepsilon^2} \sum_{(I,p,s,s')\in\mathcal{A}} a_{p,s,s'} \Big[((\bar u^\varepsilon_{I+p,s'} - \bar u^\varepsilon_{I,s}) \cdot \tau_{p,s,s'})^2 + \frac{\beta^2\varepsilon^2}{3} \Big(3(\varepsilon(\phi^\varepsilon_{I+p,s'} + \phi^\varepsilon_{I,s}) \\ - \frac{(\bar u^\varepsilon_{I+p,s'} - \bar u^\varepsilon_{I,s}) \cdot \tau^\perp_{p,s,s'}}{\ell_{p,s,s'}})^2 + (\varepsilon(\phi^\varepsilon_{I+p,s'} - \phi^\varepsilon_{I,s}))^2 \Big) \Big] + O(\sqrt{\varepsilon}) \\ \geq E_\varepsilon(\bar u^\varepsilon) + F_\varepsilon(\bar u^\varepsilon, \phi^\varepsilon) + O(\sqrt{\varepsilon}).\end{aligned}$$

Passing to the limit we get

$$\liminf \mathscr{E}_\varepsilon(u^\varepsilon) \geq \liminf \big(E_\varepsilon(\bar u^\varepsilon) + F_\varepsilon(\bar u^\varepsilon, \phi^\varepsilon) \big) \geq \inf_{U^\varepsilon,\theta^\varepsilon} \Big\{ \liminf \big(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) \big);\ U^\varepsilon \rightharpoonup u \Big\}.$$

This being true for any sequence (u^ε) with bounded energy and converging to some function u, point (i) is proven.

Now let u be a measurable vector valued function and consider any sequence $(U^\varepsilon, \theta^\varepsilon)$ with bounded energy $(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) < M)$ and such that $U^\varepsilon \rightharpoonup u$. On each thickened edge $S_{I,p,s,s'}$, Lemma 3 provides a piecewise C^1 function $u^\varepsilon_{I,p,s,s'}$ satisfying

$$u^\varepsilon_{I,p,s,s'}(x_1,x_2) = \begin{cases} U^\varepsilon_{I,s} + \theta^\varepsilon_{I,s} \times (-x_2, x_1) & \text{on } B^\varepsilon_{I,s}, \\ U^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I+p,s'} \times (-x_2, x_1) & \text{on } B^\varepsilon_{I+p,s'}, \end{cases}$$

and such that

$$\begin{aligned}\int_{S_{I,p,s,s'}} \Big(\mu \|e(u^\varepsilon_{I,p,s,s'})\|^2 + \frac{\lambda}{2} \mathrm{tr}(e(u^\varepsilon_{I,p,s,s'}))^2 \Big)\, dx \leq \frac{a_{p,s,s'}}{2\varepsilon^2} (1 + \frac{C\beta}{l_{p,s,s'}}\varepsilon) \Bigg(\big((U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \cdot \tau_{p,s,s'}\big)^2 \\ + \frac{\beta^2\varepsilon^2}{3} \Big(3(\varepsilon\theta^\varepsilon_{I+p,s'} + \varepsilon\theta^\varepsilon_{I,s}) + \frac{2}{l_{p,s,s'}} ((U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \cdot \tau^\perp_{p,s,s'}) \Big)^2 + (\varepsilon\theta^\varepsilon_{I+p,s'} - \varepsilon\theta^\varepsilon_{I,s})^2 \Bigg).\end{aligned}$$

We can now define u^ε on Ω_ε by setting $u^\varepsilon(x) := u^\varepsilon_{I,p,s,s'}(x)$ if $x \in S_{I,p,s,s'}$. Our assumptions about the geometry of the graph and our definition of k' make this definition coherent on the intersections of different thickened edges. By definition $\bar u^\varepsilon = U^\varepsilon$ and so $u^\varepsilon \rightharpoonup u$. By summation we get

$$\mathscr{E}_\varepsilon(u^\varepsilon) \leq (1 + \frac{C\beta}{l_m}\varepsilon)(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)))$$

Passing to the limit

$$\inf_{u^\varepsilon \rightharpoonup u} \limsup \mathscr{E}_\varepsilon(u^\varepsilon) \leq \limsup \mathscr{E}_\varepsilon(u^\varepsilon) \leq \limsup (E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon))).$$

This being true for any sequence (θ^ε) and any sequence (U^ε) converging to u, Point (ii) is proven. ◻

4 Main result

From now on we will seek for the Γ-limit $\mathscr{E}$ of the sequence of the discrete functionals $(E_\varepsilon + F_\varepsilon)$ defined in (9), (10).

We do not intend to study the way the different boundary conditions which could be imposed to our structures pass to the limit. That is a very interesting topic as the boundary conditions associated to second gradient material are rich and have exotic mechanical interpretation [26],[49]. But studying their whole diversity would lead to very long mathematical developments. On the other hand, as the structures we consider may present in the limit some inextensibility constraint, assuming, at it is frequent, Dirichlet boundary conditions would lead to a trivial set of admissible deformations. So we decide to consider here only free boundary conditions. As well known, in this case, the equilibrium of the structure can be reached only when the applied external actions are balanced and the solution of equilibrium problems is defined up to a global rigid motion. In order to ensure uniqueness, we need to impose that U and θ have zero mean values:

$$\sum\!\!\!\!\!\!-_I \frac{1}{K} \sum_{s=1}^K U_{I,s} = 0, \quad \sum\!\!\!\!\!\!-_I \frac{1}{K} \sum_{s=1}^K \theta_{I,s} = 0 \tag{11}$$

We associate to any sequence $(U^\varepsilon, \theta^\varepsilon)$ the families of vectors m_I^ε, $v_{I,s}^\varepsilon$ and $\chi_{I,p}^\varepsilon$ defined by

$$m_I^\varepsilon := \frac{1}{K}\sum_{s=1}^K U_{I,s}^\varepsilon, \quad v_{I,s}^\varepsilon := \frac{1}{\varepsilon}(U_{I,s}^\varepsilon - m_I^\varepsilon), \quad \chi_{p,I}^\varepsilon := \frac{1}{\varepsilon}(m_{I+p}^\varepsilon - m_I^\varepsilon) \tag{12}$$

and the family of reals $\omega_{I,p,s,s'}^\varepsilon$ defined by

$$\omega_{I,p,s,s'}^\varepsilon := \begin{cases} \varepsilon^{-2}(U_{I+p,s'}^\varepsilon - U_{I,s}^\varepsilon)\cdot \tau_{p,s,s'}, & \text{if } (p,s,s') \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Using this notation, we can rewrite the two addends of the energy, $E_\varepsilon(U^\varepsilon)$ and $F_\varepsilon(U^\varepsilon, \theta^\varepsilon)$, under the forms

$$\bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) := \varepsilon^{-2}\sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}\left((v_{I+p,s'}^\varepsilon - v_{I,s}^\varepsilon + \chi_{I,p}^\varepsilon)\cdot \tau_{p,s,s'}\right)^2 \tag{14}$$

$$\bar{F}_\varepsilon(v^\varepsilon, \chi^\varepsilon, \theta^\varepsilon) := \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}\beta^2}{6}\Big(3\big(\theta_{I+p,s'}^\varepsilon + \theta_{I,s}^\varepsilon - \frac{2}{\ell_{p,s,s'}}(v_{I+p,s'}^\varepsilon - v_{I,s}^\varepsilon + \chi_{I,p}^\varepsilon)\cdot \tau_{p,s,s'}^\perp\big)^2 + \big(\theta_{I+p,s'}^\varepsilon - \theta_{I,s}^\varepsilon\big)^2\Big). \tag{15}$$

Let us introduce the continuous counterparts of these quantities. For functions θ, v defined respectively on $\Omega \times \{1,\ldots,K\}$ and η defined on $\Omega \times \mathcal{P} \times \{1,\ldots,K\}$, square integrable with respect to their first variable and taking value respectively in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^2$, we set

$$\bar{E}(v,\eta) := \int_\Omega \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}((v_{s'}(x) - v_s(x) + \eta_{p,s'}(x))\cdot \tau_{p,s,s'})^2\,dx, \tag{16}$$

$$\bar{F}(v,\eta,\theta) := \int_\Omega \sum_{p,s,s'} \frac{a_{p,s,s'}\beta^2}{6}\Big(3\big(\theta_{s'}(x) + \theta_s(x) - \frac{2}{\ell_{p,s,s'}}(v_{s'}(x) - v_s(x) + \eta_{p,s'}(x))\cdot \tau_{p,s,s'}^\perp\big)^2 + \big(\theta_{s'}(x) - \theta_s(x)\big)^2\Big). \tag{17}$$

We extend this definition to distributions by setting $\bar{E} = +\infty$ or $\bar{F} = +\infty$ whenever the integrands are not square integrable. For any functions u and v respectively in $\mathrm{L}^2(\mathbb{R}^2)$ and $\mathrm{L}^2(\mathbb{R}^2 \times \{1,\ldots,K\})$ we set, in the sense of distributions, for any $(p,s) \in \mathcal{P} \times \{1,\ldots,K\}$

$$(\eta_u)_{p,s} := \nabla u \cdot \mathbf{p}, \tag{18}$$

$$(\xi_{u,v})_{p,s} = \nabla v_s \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u \cdot \mathbf{p}\cdot \mathbf{p}. \tag{19}$$

The limit energy of our structure reads

$$\mathscr{E}(u) := \inf_{w,v,\theta}\left\{\bar{E}(w, \xi_{u,v}) + \bar{F}(v, \eta_u, \theta);\ \bar{E}(v,\eta_u) = 0\right\}. \tag{20}$$

Indeed we have

Theorem 2. *The sequence $(E_\varepsilon + F_\varepsilon)$ Γ-converges to $\mathscr{E}$:*
(i) For all sequence $(U^\varepsilon, \theta^\varepsilon)$ such that $U^\varepsilon \rightharpoonup u$, we have $\liminf(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)) \geq \mathscr{E}(u)$.
(ii) For any u such that $\mathscr{E}(u) < +\infty$, there exists a sequence $(U^\varepsilon, \theta^\varepsilon)$ such that $U^\varepsilon \rightharpoonup u$ and $\limsup(E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)) \leq \mathscr{E}(u)$.

In the next three sections we first prove the relative compactness of the sequences $(U^\varepsilon, \theta^\varepsilon)$ with bounded energies and of the associated sequences m^ε, v^ε, χ^ε; then we establish relationships between the limits of these quantities and finally we prove Theorem 2.

4.1 Compactness

Lemma 4. *Let $(U^\varepsilon, \theta^\varepsilon)$ with zero mean rigid motion satisfying $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) \leq M$, then the sequences $(\sum_I \|U_{I,s}^\varepsilon\|^2)$, $(\sum_I (\theta_{I,s}^\varepsilon)^2)$, $(\sum_I \|m_I^\varepsilon\|^2)$, $(\sum_I \|v_{I,s}^\varepsilon\|^2)$, $(\sum_I \|\chi_{I,p}^\varepsilon\|^2)$ and $(\sum_I (\omega_{I,p,s,s'}^\varepsilon)^2)$ are bounded.*

Proof. The proof is based on the connectedness assumption, on successive applications of triangle inequality and on the classical Korn inequality. Here M is a constant which can change from line to line.

Consider (p, s, s') and (q, s', s'') in $\mathcal{A}$. From expression (9) of E_ε we immediately deduce that

$$\sum_I (\omega^\varepsilon_{I,p,s,s'})^2 = \varepsilon^{-2} \sum_I \left(\frac{U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}}{\varepsilon} \cdot \tau_{p,s,s'} \right)^2 < M. \tag{21}$$

From expression (10) of F_ε we also deduce that

$$\sum_I \left(\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s}\right)^2 < M \quad \text{and} \quad \sum_I \left(\frac{U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}}{\varepsilon \ell_{p,s,s'}} \cdot \tau^\perp_{p,s,s'} - \theta^\varepsilon_{I,s} \right)^2 < M.$$

Owing to our connectedness assumption, let us introduce a path $(s_i, p_i, \epsilon_i)_{i=1}^r$ connecting node $y^\varepsilon_{I,s}$ to node $y^\varepsilon_{I+p,s'}$ as described in section 2.1. For $0 \le j \le r$, we set $\tilde{\mathbf{p}}_j := \sum_{i=1}^{j-1} \epsilon_i \mathbf{p}_i$. Using triangle inequality we have

$$\sum_I \left(\theta^\varepsilon_{I+\tilde{p}_j,s_j} - \theta^\varepsilon_{I,s}\right)^2 < M \quad \text{and thus} \quad \sum_I \left(\epsilon_j \frac{U^\varepsilon_{I+\tilde{p}_{j+1},s_{j+1}} - U^\varepsilon_{I+\tilde{p}_{j+1},s_j}}{\varepsilon \ell_{p_j,s_j,s_{j+1}}} \cdot \tau^\perp_{p_j,s_j,s_{j+1}} - \theta^\varepsilon_{I,s} \right)^2 < M.$$

Setting $\tilde{U}^\varepsilon_{I,s,J,s'} := U^\varepsilon_{J,s'} - \theta^\varepsilon_{I,s}(y^\varepsilon_{J,s'} - y^\varepsilon_{I,s})^\perp$ so that

$$\tilde{U}^\varepsilon_{I,s,J+p,s''} - \tilde{U}^\varepsilon_{I,s,J,s'} = U^\varepsilon_{J+p,s''} - U^\varepsilon_{J,s'} - \varepsilon \theta^\varepsilon_{I,s} \ell_{p,s',s''} \tau^\perp_{p,s',s''},$$

last inequality reads

$$\sum_I \left(\frac{\tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_{j+1}} - \tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_j}}{\varepsilon \ell_{p_j,s_j,s_{j+1}}} \cdot \tau^\perp_{p_j,s_j,s_{j+1}} \right)^2 < M.$$

As (21) also implies $\varepsilon^{-2} \sum_I \left((\tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_{j+1}} - \tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_j}) \cdot \tau_{p_j,s_j,s_{j+1}} \right)^2 < M$, we get

$$\varepsilon^{-2} \sum_I \left\| \tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_{j+1}} - \tilde{U}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_j} \right\|^2 < M$$

which leads, still using triangle inequality, to $\varepsilon^{-2} \sum_I \left\| \tilde{U}^\varepsilon_{I,s,I+p,s'} - \tilde{U}^\varepsilon_{I,s,I,s} \right\|^2 < M$ or equivalently

$$\varepsilon^{-2} \sum_I \left\| U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s} - \theta^\varepsilon_{I,s}(y^\varepsilon_{I+p,s'} - y^\varepsilon_{I,s})^\perp \right\|^2 < M. \tag{22}$$

We focus temporarily on the particular case $s = s' = 1$ which reads

$$\varepsilon^{-2} \sum_I \left\| U^\varepsilon_{I+p,1} - U^\varepsilon_{I,1} - \varepsilon \theta^\varepsilon_{I,1} \mathbf{p}^\perp \right\|^2 < M \tag{23}$$

and, in order to pass from this local rigidity inequality to a global one without stating any discrete version of Korn inequality, we use a H^1 interpolation. We fix two independent vectors p and q in $\mathcal{P}$ and divide the domain in the disjoint union (for all I) of triangles $(y^\varepsilon_{I,1}, y^\varepsilon_{I+p,1}, y^\varepsilon_{I+p+q,1})$ and $(y^\varepsilon_{I,1}, y^\varepsilon_{I+q,1}, y^\varepsilon_{I+p+q,1})$. The piecewise affine interpolation $\mathcal{U}^\varepsilon$ of U^ε defined by setting for any I,

$$\mathcal{U}^\varepsilon\left(y^\varepsilon_{I,1} + \varepsilon(ap + bq)\right) := \begin{cases} (1-a)\, U^\varepsilon_{I,1} + (a-b)\, U^\varepsilon_{I+p,1} + b\, U^\varepsilon_{I+p+q,1} & \text{if } 0 \le b \le a \le 1, \\ (1-b)\, U^\varepsilon_{I,1} + (b-a)\, U^\varepsilon_{I+q,1} + a\, U^\varepsilon_{I+p+q,1} & \text{if } 0 \le a \le b \le 1, \end{cases}$$

belongs to H^1 and satisfies $\|\mathcal{U}^\varepsilon\|^2_{\mathrm{L}^2} \ge C \sum_I \|U^\varepsilon_{I,1}\|^2$ for some constant C. Moreover on the two types of triangles we have either $\varepsilon \nabla \mathcal{U}^\varepsilon \cdot \mathbf{p} = U^\varepsilon_{I+p,1} - U^\varepsilon_{I,1}$ and $\varepsilon \nabla \mathcal{U}^\varepsilon \cdot \mathbf{q} = U^\varepsilon_{I+p+q,1} - U^\varepsilon_{I+p,1}$ or $\varepsilon \nabla \mathcal{U}^\varepsilon \cdot \mathbf{p} = U^\varepsilon_{I+p+q,1} - U^\varepsilon_{I+q,1}$ and $\varepsilon \nabla \mathcal{U}^\varepsilon \cdot \mathbf{q} = U^\varepsilon_{I+q,1} - U^\varepsilon_{I,1}$. In the first case we can write

$$\begin{aligned}
\varepsilon\, e(\mathcal{U}^\varepsilon) | (\mathbf{p} \otimes \mathbf{p}) &= (U^\varepsilon_{I+p,1} - U^\varepsilon_{I,1} - \varepsilon \theta^\varepsilon_{I,1} \mathbf{p}^\perp) \cdot \mathbf{p} \\
\varepsilon\, e(\mathcal{U}^\varepsilon) | (\mathbf{q} \otimes \mathbf{q}) &= (U^\varepsilon_{I+p+q,1} - U^\varepsilon_{I,1} - \varepsilon \theta^\varepsilon_{I+p,1} \mathbf{q}^\perp) \cdot \mathbf{q} \\
\varepsilon\, e(\mathcal{U}^\varepsilon) | (\mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}) &= (U^\varepsilon_{I+p,1} - U^\varepsilon_{I,1} - \varepsilon \theta^\varepsilon_{I,1} \mathbf{p}^\perp) \cdot \mathbf{q} + (U^\varepsilon_{I+p+q,1} - U^\varepsilon_{I,1} - \varepsilon \theta^\varepsilon_{I+p,1} \mathbf{q}^\perp) \cdot \mathbf{p}
\end{aligned}$$

where we have used the identity $\mathbf{p}^\perp\cdot\mathbf{q}+\mathbf{q}^\perp\cdot\mathbf{p}=0$. The second case is similar. As $\mathbf{p}\otimes\mathbf{p}$, $\mathbf{q}\otimes\mathbf{q}$, $(\mathbf{p}\otimes\mathbf{q}+\mathbf{q}\otimes\mathbf{p})$ make a basis for symmetric matrices, we obtain $\|e(\mathcal{U}^\varepsilon)\|_{\mathrm{L}^2}^2\le M$ by using (23) and summing over all triangles. We can then use the classical Korn inequality : up to a global rigid motion, the function $\mathcal{U}^\varepsilon$ satisfies $\|\nabla\mathcal{U}^\varepsilon\|_{\mathrm{L}^2}^2\le M$ and owing to Poincaré inequality $\|\mathcal{U}^\varepsilon\|_{\mathrm{L}^2}^2\le M$. From $\|\mathcal{U}^\varepsilon\|_{\mathrm{L}^2}^2\le M$ we directly get $\sum_I\|U_{I,1}^\varepsilon\|^2<M$. From $\|\nabla\mathcal{U}^\varepsilon\cdot\mathbf{p}\|_{\mathrm{L}^2}^2\le M$ we get $\varepsilon^{-2}\sum_I\left\|U_{I+p,1}^\varepsilon-U_{I,1}^\varepsilon\right\|^2<M$ and thus $\sum_I(\theta_{I,s}^\varepsilon)^2<M$.

We can now go back to (22). We get

$$\varepsilon^{-2}\sum_I\left\|U_{I+p,s'}^\varepsilon-U_{I,s}^\varepsilon\right\|^2<M. \tag{24}$$

In particular $\varepsilon^{-2}\sum_I\left\|U_{I,s}^\varepsilon-U_{I,1}^\varepsilon\right\|^2<M$ and so

$$\sum_I\left\|U_{I,s}^\varepsilon\right\|^2<M \tag{25}$$

Other bounds are now easy to get : taking the mean value with respect to s in (25) and (24) (with $p=0$) gives respectively

$$\sum_I\|m_I^\varepsilon\|^2\le M,\quad \sum_I\|v_{I,s}^\varepsilon\|^2\le M, \tag{26}$$

while taking the mean value with respect to s and s' in (24) gives

$$\sum_I\|\chi_{I,p}^\varepsilon\|^2\le M. \tag{27}$$

□

4.2 Double scale convergence

The bounds that we established in Lemma 4 imply the existence of θ, u, v, χ_p and $\omega_{p,s,s'}$ in L^2 such that, for any s and up to subsequences,

$$\theta_s^\varepsilon\rightharpoonup\theta_s,\quad m^\varepsilon\rightharpoonup u,\quad v_s^\varepsilon\rightharpoonup v_s,\quad \chi_p^\varepsilon\rightharpoonup\chi_p\quad\text{and}\quad \omega_{p,s,s'}^\varepsilon\rightharpoonup\omega_{p,s,s'}. \tag{28}$$

The following lemma establishes useful properties of these limits. We follow the methods used in [4] for establishing properties of double limits.

Lemma 5. *We have*

$$U_s^\varepsilon\rightharpoonup u,\qquad \sum_{s=1}^K v_s=0\quad\textit{and}\qquad \chi_p=\nabla u\cdot\mathbf{p}. \tag{29}$$

Moreover there exist some fields w_s and λ in $\mathrm{L}^2(\mathbb{R}^2)$ such that, for any $(p,s,s')\in\mathcal{A}$,

$$\omega_{p,s,s'}=\big(w_{s'}-w_s+\nabla(v_{s'}+\lambda)\cdot\mathbf{p}+\frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p}\big)\cdot\tau_{p,s,s'} \tag{30}$$

Proof. The convergence of v_s^ε implies that $(U_s^\varepsilon-m^\varepsilon)\rightharpoonup 0$ and so $U_s^\varepsilon\rightharpoonup u$.

The fact that $\sum_{s=1}^K v_{I,s}^\varepsilon=0$ clearly implies that $\sum_{s=1}^K v_s(x)=0$.

To check that $\chi_p=\nabla u\cdot\mathbf{p}$, it is enough to notice that, for any smooth test field φ with compact support,

$$\begin{aligned}\int_\Omega\chi_p(x)\cdot\varphi(x)&=\lim\sum_I\varepsilon^{-1}(m_{I+p}^\varepsilon-m_I^\varepsilon)\cdot\varphi(y_I^\varepsilon)=\lim\sum_I m_I^\varepsilon\cdot\varepsilon^{-1}(\varphi(y_{I-p}^\varepsilon)-\varphi(y_I^\varepsilon))\\&=\lim\sum_I m_I^\varepsilon\cdot\big(-\nabla\varphi(y_I^\varepsilon)\cdot\mathbf{p})+O(\varepsilon)\big)=-\int_\Omega u(x)\cdot\big(\nabla\varphi(x)\cdot p\big)=\int_\Omega\big(\nabla u(x)\cdot p\big)\cdot\varphi(x).\end{aligned}$$

Characterizing the limit $\omega_{p,s,s'}$ is more delicate. To that aim, let us introduce the set $\mathcal{D}_A$ of families of distributions in $H^{-1}(\mathbb{R}^2)$:

$$\mathcal{D}_A:=\big\{\psi_{p,s,s'}=(w_s'-w_s+\nabla\lambda\cdot p)\cdot\tau_{p,s,s'}:\ (p,s,s')\in\mathcal{A},\ w_s\in\mathrm{L}^2,\ \lambda\in\mathrm{L}^2\big\}$$

and ${\mathcal{D}_A}^{\perp}$ its orthogonal, that is the set of families $(\phi_{p,s,s'})_{(p,s,s')\in\mathcal{A}}$ of functions in $H^1(\mathbb{R}^2)$ such that, for all $\psi_{p,s,s'}\in\mathcal{D}_A$, $\sum_{(p,s,s')\in\mathcal{A}}\langle\psi_{p,s,s'},\phi_{p,s,s'}\rangle=0$. Let us remark that, for any $\phi\in{\mathcal{D}_A}^{\perp}$ we have

$$\sum_{(p,s,s')\in\mathcal{A}}(\nabla\phi_{p,s,s'}\cdot\mathbf{p})\,\tau_{p,s,s'}=0. \tag{31}$$

and for any $(w_s)\in\mathrm{L}^2(\mathbb{R}^2,\mathbb{R}^2)^K$,

$$\sum_{(p,s,s')\in\mathcal{A}}((w_{s'}-w_s)\cdot\tau_{p,s,s'})\,\phi_{p,s,s'}=0. \tag{32}$$

If we extend ϕ by setting $\phi_{p,s,s'}=0$ whenever $(p,s,s')\notin\mathcal{A}$ we can rewrite this last equation as

$$\sum_{(p,s,s')}\tau_{p,s,s'}\,\phi_{p,s,s'}-\tau_{p,s',s}\,\phi_{p,s',s}=0.$$

Thus for such functions we have, using (32),

$$\begin{aligned}\int_\Omega\sum_{(p,s,s')\in\mathcal{A}}\omega_{p,s,s'}(x)\phi_{p,s,s'}(x)&=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}\varepsilon^{-2}(U^\varepsilon_{I+p,s'}-U^\varepsilon_{I,s})\cdot(\phi_{p,s,s'}(y^\varepsilon_I)\tau_{p,s,s'})\\ =\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}&(\varepsilon^{-1}(v^\varepsilon_{I+p,s'}-v^\varepsilon_{I,s'})+\varepsilon^{-1}(v^\varepsilon_{I,s'}-v^\varepsilon_{I,s})+\varepsilon^{-2}(m^\varepsilon_{I+p}-m^\varepsilon_I))\cdot(\phi_{p,s,s'}(y^\varepsilon_I)\tau_{p,s,s'})\\ =\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}&(\varepsilon^{-1}(v^\varepsilon_{I+p,s'}-v^\varepsilon_{I,s'})+\varepsilon^{-2}(m^\varepsilon_{I+p}-m^\varepsilon_I))\cdot(\phi_{p,s,s'}(y^\varepsilon_I)\tau_{p,s,s'}).\end{aligned}$$

Considering only smooth functions $\phi_{p,s,s'}$ with compact support we can estimate the first addend by

$$\begin{aligned}&\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}\varepsilon^{-1}(v^\varepsilon_{I+p,s'}-v^\varepsilon_{I,s'})\cdot(\phi_{p,s,s'}(y^\varepsilon_I)\tau_{p,s,s'})\\ &\qquad=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}v^\varepsilon_{I,s'}\cdot(\varepsilon^{-1}(\phi_{p,s,s'}(y^\varepsilon_{I-p})-\phi_{p,s,s'}(y^\varepsilon_I))\tau_{p,s,s'})\\ &\qquad=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}v^\varepsilon_{I,s'}\cdot((-\nabla\phi_{p,s,s'}(y^\varepsilon_I)\cdot\mathbf{p})\tau_{p,s,s'})+O(\varepsilon)\\ &\qquad=-\int_\Omega\sum_{(p,s,s')\in\mathcal{A}}v_{s'}(x)\cdot((\nabla\phi_{p,s,s'}(x)\cdot\mathbf{p})\tau_{p,s,s'})\,dx\\ &\qquad=\sum_{(p,s,s')\in\mathcal{A}}\Big\langle(\nabla v_{s'}(x)\cdot\mathbf{p}),(\phi_{p,s,s'}(x)\tau_{p,s,s'})\Big\rangle.\end{aligned}$$

The second addend becomes using (31)

$$\begin{aligned}&\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}\varepsilon^{-2}(m^\varepsilon_{I+p}-m^\varepsilon_I))\cdot(\phi_{p,s,s'}(y^\varepsilon_I)\tau_{p,s,s'})\\ &=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}\varepsilon^{-2}m^\varepsilon_I\cdot((\phi_{p,s,s'}(y^\varepsilon_{I-p})-\phi_{p,s,s'}(y^\varepsilon_I))\tau_{p,s,s'})\\ &=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}m^\varepsilon_I\cdot((-\varepsilon^{-1}\nabla\phi_{p,s,s'}(y^\varepsilon_I)\cdot\mathbf{p}+\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(y^\varepsilon_I)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'})\\ &=\lim\mathop{\sum\!\!\!\!\!\!-}_{I}\sum_{(p,s,s')\in\mathcal{A}}m^\varepsilon_I\cdot((\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(y^\varepsilon_I)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'})\\ &=\int_\Omega\sum_{(p,s,s')\in\mathcal{A}}u(x)\cdot((\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(x)\cdot\mathbf{p}\cdot\mathbf{p})\tau_{p,s,s'})\,dx\\ &=\sum_{(p,s,s')\in\mathcal{A}}\frac{1}{2}\Big\langle(\nabla\nabla u(x)\cdot\mathbf{p}\cdot\mathbf{p}),(\phi_{p,s,s'}(x)\tau_{p,s,s'})\Big\rangle.\end{aligned}$$

Collecting these results we obtain that the distribution

$$\omega_{p,s,s'} - \big(\nabla v_{s'} \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u \cdot \mathbf{p} \cdot \mathbf{p}\big) \cdot \tau_{p,s,s'}$$

is orthogonal to all smooth functions in ${\mathcal{D}_A}^\perp$ with compact support in Ω. As they are dense in ${\mathcal{D}_A}^\perp$, we know that there exist some fields w_s and λ in $\mathrm{L}^2(\mathbb{R}^2)$ such that, for any $(p,s,s') \in \mathcal{A}$,

$$\omega^p_{s,s'} = \big(w_{s'} - w_s + \nabla(v_{s'} + \lambda) \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u \cdot \mathbf{p} \cdot \mathbf{p}\big) \cdot \tau_{p,s,s'}.$$

□

4.3 Proof of the homogenization result

Proof. To prove assertion (i) of Theorem 2, we consider a sequence $(U^\varepsilon, \theta^\varepsilon)$ with bounded energy : $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon) \le M$ (otherwise the result is trivial). Therefore $\varepsilon^2 E_\varepsilon(U^\varepsilon)$ tends to zero. We know from (28) and (29) and Remark 1 that $v^\varepsilon_s \rightharpoonup v_s$ and $\chi^\varepsilon_p \rightharpoonup \eta_u$. From Remark 2 we get

$$0 = \liminf_\varepsilon \big(\varepsilon^2 E_\varepsilon(U^\varepsilon)\big) = \liminf_\varepsilon \big(\varepsilon^2 \bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon)\big) \ge \bar{E}(v, \eta_u).$$

Hence $\bar{E}(v,\eta_u) = 0$. Rewriting now $E_\varepsilon(U^\varepsilon)$ as $\sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2}(\omega^\varepsilon_{I,p,s,s'})^2$, the energy reads

$$\begin{aligned}\bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) + \bar{F}_\varepsilon(v^\varepsilon, \chi^\varepsilon, \theta^\varepsilon) = \sum_I \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2} \Bigg(\big(\omega^\varepsilon_{I,p,s,s'}\big)^2 + \frac{\beta^2}{3}\Big(3\big(\theta^\varepsilon_{I+p,s'} + \theta^\varepsilon_{I,s} \\ - \frac{2}{\ell_{p,s,s'}}(v^\varepsilon_{I+p,s'} - v^\varepsilon_{I,s} + \chi^\varepsilon_{I,p}) \cdot \tau^\perp_{p,s,s'}\big)^2 + \big(\theta^\varepsilon_{I+p,s'} - \theta^\varepsilon_{I,s}\big)^2\Big)\Bigg) \qquad (33)\end{aligned}$$

Using again (28), (29), (30) and Remarks 1 and 2 we get

$$\begin{aligned}\liminf_\varepsilon \big(\bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) + \bar{F}_\varepsilon(v^\varepsilon, \chi^\varepsilon, \theta^\varepsilon)\big) \ge \int_\Omega \sum_{(p,s,s')} \frac{a_{p,s,s'}}{2} \Bigg(\big(\omega_{p,s,s'}(x)\big)^2 + \frac{\beta^2}{3}\Big(3\big(\theta_{s'}(x) + \theta_s(x) \\ - \frac{2}{\ell_{p,s,s'}}(v_{s'}(x) - v_s(x) + \chi_p(x)) \cdot \tau^\perp_{p,s,s'}\big)^2 + \big(\theta_{s'}(x) - \theta_s(x)\big)^2\Big)\Bigg)\,dx \\ \ge \bar{E}(w, \xi_{u,v+\lambda}) + \bar{F}(v, \eta_u, \theta). \qquad (34)\end{aligned}$$

Noticing that $\bar{F}(v-\lambda, \eta_u, \theta) = \bar{F}(v, \eta_u, \theta)$ and $\bar{E}(v+\lambda, \eta_u) = \bar{E}(v,\eta_u) = 0$, we get the desired bound.

In order to prove assertion (ii), let us now consider a function u such that $\mathcal{E}(u) < +\infty$. By a density argument, we can assume that $u \in C^\infty(\Omega)$. We introduce (v, w, θ) such that $\mathcal{E}(u) = \bar{E}(w, \xi_{u,v}) + \bar{F}(v, \eta_u, \theta)$ and $\bar{E}(v, \eta_u) = 0$. Their existence is ensured by the coercivity and lower semicontinuity of these functionals. The fields (v, w, θ) also belong to $C^\infty(\Omega)$. Note that $\bar{E}(v,\eta_u) = 0$ implies that, for any $(p,s,s') \in \mathcal{A}$,

$$(v_{s'} - v_s + \nabla u \cdot \mathbf{p}) \cdot \tau_{p,s,s'} = 0 \qquad (35)$$

from which we can deduce that

$$(\nabla v_{s'} \cdot \mathbf{p} - \nabla v_s \cdot \mathbf{p} + \nabla\nabla u \cdot \mathbf{p} \cdot \mathbf{p}) \cdot \tau_{p,s,s'} = 0. \qquad (36)$$

We now define U^ε and θ^ε by setting

$$U^\varepsilon_{I,s} := u(y^\varepsilon_I) + \varepsilon v_s(y^\varepsilon_I) + \varepsilon^2 w_s(y^\varepsilon_I) \quad \text{and} \quad \theta^\varepsilon_{I,s} := \theta_s(y^\varepsilon_I). \qquad (37)$$

It is clear that $U^\varepsilon \rightharpoonup u$ and $\theta_s^\varepsilon \rightharpoonup \theta_s$. Let us compute $E_\varepsilon(U^\varepsilon) + F_\varepsilon(U^\varepsilon, \theta^\varepsilon)$. We have, using (35) and (36),

$$\begin{aligned}
\omega^\varepsilon_{I,p,s,s'} &= \varepsilon^{-2}\tau_{p,s,s'} \cdot (U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) \\
&= \tau_{p,s,s'} \cdot \Big(\varepsilon^{-2}(u(y^\varepsilon_{I+p}) - u(y^\varepsilon_I)) + w_{s'}(y^\varepsilon_{I+p}) - w_s(y^\varepsilon_I) \\
&\qquad\qquad + \varepsilon^{-1}(v_{s'}(y^\varepsilon_{I+p}) - v_s(y^\varepsilon_{I+p})) + \varepsilon^{-1}(v_s(y^\varepsilon_{I+p}) - v_s(y^\varepsilon_I))\Big) \\
&= \tau_{p,s,s'} \cdot \Big(\varepsilon^{-1}\nabla u(y^\varepsilon_I)\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u(y^\varepsilon_I)\cdot\mathbf{p}\cdot\mathbf{p} + w_{s'}(y^\varepsilon_{I+p}) - w_s(y^\varepsilon_I) \\
&\qquad\qquad - \varepsilon^{-1}\nabla u(y^\varepsilon_{I+p})\cdot\mathbf{p} + \nabla v_s(y^\varepsilon_I)\cdot\mathbf{p}\Big) + O(\varepsilon) \\
&= \tau_{p,s,s'} \cdot \Big(-\frac{1}{2}\nabla\nabla u(y^\varepsilon_I)\cdot\mathbf{p}\cdot\mathbf{p} + \nabla v_s(y^\varepsilon_I)\cdot\mathbf{p} + w_{s'}(y^\varepsilon_{I+p}) - w_s(y^\varepsilon_I)\Big) + O(\varepsilon) \\
&= \tau_{p,s,s'} \cdot \Big(\frac{1}{2}\nabla\nabla u(y^\varepsilon_I)\cdot\mathbf{p}\cdot\mathbf{p} + \nabla v_{s'}(y^\varepsilon_I)\cdot\mathbf{p} + w_{s'}(y^\varepsilon_{I+p}) - w_s(y^\varepsilon_I)\Big) + O(\varepsilon).
\end{aligned}$$

Hence $\omega^\varepsilon_{I,p,s,s'} = \tau_{p,s,s'} \cdot \Big(w_{s'}(y^\varepsilon_{I+p}) - w_s(y^\varepsilon_I) + (\xi_{u,v})_{p,s'}(y^\varepsilon_I)\Big) + O(\varepsilon)$ and

$$\begin{aligned}
\lim E_\varepsilon(U^\varepsilon) &= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a^p_{s,s'}}{2}(\omega^\varepsilon_{I,p,s,s'})^2 \\
&= \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \frac{a^p_{s,s'}}{2}\Big(\big(w_{s'}(x) - w_s(x) + (\xi_{u,v})_{p,s,s'}\big)\cdot\tau_{p,s,s'}\Big)^2 = \bar{E}(w, \xi_{u,v}).
\end{aligned} \tag{38}$$

On the other hand

$$\begin{aligned}
\varepsilon^{-1}\tau^\perp_{p,s,s'} \cdot (U^\varepsilon_{I+p,s'} - U^\varepsilon_{I,s}) &= \tau^\perp_{p,s,s'} \cdot \Big(\varepsilon^{-1}\big(u(y^\varepsilon_{I+p}) - u(y^\varepsilon_I)\big) + v_{s'}(y^\varepsilon_{I+p}) - v_s(y^\varepsilon_I)\Big) + O(\varepsilon) \\
&= \tau^\perp_{p,s,s'} \cdot \Big(\nabla u(y^\varepsilon_I)\cdot\mathbf{p} + v_{s'}(y^\varepsilon_{I+p}) - v_s(y^\varepsilon_I)\Big) + O(\varepsilon).
\end{aligned}$$

As $v_s(y^\varepsilon_{I+p}) = v_s(y^\varepsilon_I) + O(\varepsilon)$ and $\theta_s(y^\varepsilon_{I+p}) = \theta_s(y^\varepsilon_I) + O(\varepsilon)$, we have

$$\begin{aligned}
\lim F_\varepsilon(U^\varepsilon, \theta^\varepsilon) &= \lim \sum\!\!\!\!\!\!\int_I \sum_{(p,s,s')} \frac{a_{p,s,s'}\beta^2}{6}\Big(3\big(\theta_{s'}(y^\varepsilon_I) + \theta_s(y^\varepsilon_I) \\
&\qquad - \frac{2}{\ell_{p,s,s'}}(v_{s'}(y^\varepsilon_I) - v_s(y^\varepsilon_I) + \nabla u(y^\varepsilon_I)\cdot\mathbf{p})\cdot\tau^\perp_{p,s,s'}\big)^2 + \big(\theta_{s'}(y^\varepsilon_I) - \theta_s(y^\varepsilon_I)\big)^2\Big) \\
&= \bar{F}(v, \eta_u, \theta).
\end{aligned} \tag{39}$$

The result is obtained by collecting (38) and (39). □

5 Making explicit the limit energy

In the limit energy we have identified, namely

$$\mathscr{E}(u) := \inf_{w,v,\theta}\left\{\bar{E}(w,\xi_{u,v}) + \bar{F}(v,\eta_u,\theta);\ \bar{E}(v,\eta_u) = 0\right\}.$$

one has to compute the minimum with respect to three extra kinematic variables. These minima can essentially be computed locally, through "a cell problem". This is clearly the case for θ and w for which solutions depend linearly respectively on $\xi_{u,v}$ and (v,η_u). The quadratic constraint $E(v,\eta_u) = 0$ is also easily solved and v takes the form $v = L\cdot\eta_u + \lambda$ with L a linear operator and λ any field in the kernel of this energy. Collecting these results, $\mathscr{E}$ becomes the integral of a quadratic form of the quantities ∇u, $\nabla\nabla u$, λ and $\nabla\lambda$. This procedure is pure linear algebra dealing with very low dimensional matrices. The computation can even be performed analytically but using a software like Octave© or Matlab® saves a lot of time.

A priori, the infimum with respect to λ cannot be computed locally and the limit model involves this extra kinematic variable : it is both a generalized continuum model [29] and a strain gradient model. The variable λ, which could be called a "micro-adjustment", always play a fundamental role in the limit energy. However in many cases, $\nabla\lambda$ can be computed locally and λ can be eliminated. This is the case for the three examples we have provided in figures 1 and 3. For making explicit the limit energy of these three examples, we fix the values of the interactions by assuming that $a_{p,s,s'} = 1$ for all connecting edges and that $\beta = 1$: we obtain

- the homogenized material corresponding to graph given in Figure 3a is submitted to the constraints

$$e_{1,1}(u) = 0, \qquad e_{2,2}(u) = 0$$

and has the following elastic energy

$$\mathscr{E}(u) = \int_\Omega 3(e_{1,2}(u))^2\, dx.$$

It corresponds to a classical elastic material which is inextensible in the directions e_1 and e_2.

- the homogenized material corresponding to graph given in Figure 3b is submitted to the same constraints $e_{1,1}(u) = 0$, $e_{2,2}(u) = 0$ but has the following elastic energy

$$\mathscr{E}(u) = \int_\Omega \frac{96}{5}\big(e_{1,2}(u)\big)^2 + \frac{1}{16}\Big(\frac{\partial^2 u_2}{\partial x_1^2}\Big)^2\, dx.$$

We get here a second gradient material. The point is that all odd horizontal layers of the structure, owing to the diagonal bars, behave like bending beams. The second gradient term of the homogenized energy results from this phenomenon. We are in a similar case as the ones described in [44], [19]. The energy is not complete : indeed it can be rewritten

$$\mathscr{E}(u) = \int_\Omega \frac{96}{5}\big(e_{1,2}(u)\big)^2 + \frac{1}{16}\Big(\frac{\partial}{\partial x_1}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)\Big)^2\, dx.$$

The homogenized material enters in the framework of couple-stress models.

- the homogenized material corresponding to pantographic graph given in Figure 1 is more interesting. It is submitted to the constraint $e_{2,2}(u) = 0$ and has the following elastic energy

$$\mathscr{E}(u) = \int_\Omega \left(72\big(e_{1,1}(u)\big)^2 + 144\big(e_{1,2}(u)\big)^2 + \frac{3}{88}\left(\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + \frac{1}{524}\left(44\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + 13\frac{\partial^2 u_2}{\partial x_1^2}\right)^2\right)\right) dx_1 dx_2$$

The presence of the term $\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2$ makes this energy a complete second gradient one. Due to this term, the homogenized material has an exotic behavior : when a part of the domain is extended in the e_1 direction, then this extension tends to expand in the e_1 direction. The term $(e_{1,1})^2$ damps this expansion with a characteristic length $\sqrt{\frac{3}{88\times 72}} \approx 0.13$.

6 Conclusion

We have proved that the homogenized energies of our graph-based structures correspond to second gradient models possibly with an extra kinematic variable. Using our general result, it is very easy to test different designs and to understand the source of second gradient effects. Our results differ from the energies which could be obtained by a formal asymptotic development (for instance in [42] equation (4.46)). It is obvious that one must not approximate the displacement field inside the structure by a Taylor expansion of the macroscopic displacement field : this approximation is much too rough for our structures and would overestimate the homogenized energy. But assuming a double scale expansion for the displacement field is still not accurate enough: such an expansion cannot describe the flexural behavior of the slender rectangles which (see the proofs of lemmas 2 and 3 in the appendix) is due to Bernoulli-Navier displacements at scale ε^2. However, even if a double scale expansion cannot be used for our original problem (3), we note in view of (37), that it could be used for the displacements of the nodes in the discrete problem $E_\varepsilon + F_\varepsilon$ and would lead to the correct homogenized behavior.

We must warn the reader that the source of second gradient effects does not ensue from flexural interactions. The fact that bending stiffness is by itself a second gradient effect may indeed be misleading. The reader may infer that the presence in our structures of slender slabs, in which Euler-Bernoulli-Navier motions take place, is the source of the homogenized second gradient effects. We emphasize that is not the case : even when $\beta = 0$, that is when bending stiffness is neglected, second gradient effects remain. They are due to the extensional stiffness of the slender slabs and to a particular design of the periodic cell while the bending stiffness is, on the contrary, the source of the first gradient effects in the homogenized energy and ensures the relative compactness of the considered energies. To understand the nature of the appearance of second gradient through the homogenization process and to conceive new structures with such effects, we recommend the reader to forget bending stiffness and focus on the case $\beta = 0$ reminding that relative compactness could also be ensured by suitable boundary conditions.

The reader must also be aware that, depending on the geometry of the considered graph, second gradient effects may be present or not. They are present when considering the graphs represented in Figure 3b or 1 but absent when considering the graph of figure 3a. The interesting designs are those having floppy modes when considering only the extensional energy but such that the floppy modes must have uniform strain. For instance, the interest of the graph represented in Figure 1 lies in its uniform horizontal global extension floppy mode.

The detailed description of the algorithm which makes explicit the limit energy for different dimensions of space and periodicity will be given in [1].

7 Appendix

Here we collect sketches of the technical proofs of the lemma needed for the reduction of the 2D elastic problem to the discrete one.

Proof of lowerbound Lemma 2: By adding if needed a rigid motion to u, we can restrict our attention to the case $U_2^- = U_2^+ = 0$ and $U_1^- = -U_1^+$. We also remark that, for any $\sigma \in \mathrm{L}^2(\omega)$,

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \geq - \int_\omega \left(\frac{1}{4\tilde{\mu}} \|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})} \mathrm{tr}(\sigma)^2 \right) + \int_\omega \sigma : e(u).$$

Let us choose

$$\sigma = \begin{pmatrix} a + 2b(x_1 + c)x_2 & b(e^2 - x_2^2) \\ b(e^2 - x_2^2) & 0 \end{pmatrix}$$

with $a = \frac{2Y}{\ell} U_1^+$, $b = -\frac{3Y}{\ell^2}(\theta^- + \theta^+)$ and $c = \frac{\ell}{6} \frac{\theta^+ - \theta^-}{\theta^- + \theta^+}$. Setting $\tilde{Y}(x) := \frac{4\tilde{\mu}(\tilde{\mu} + \tilde{\lambda})}{2\tilde{\mu} + \tilde{\lambda}}$ (which takes the values Y and kY), we have

$$\frac{1}{4\tilde{\mu}} \|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})} \mathrm{tr}(\sigma)^2 = \frac{1}{2\tilde{Y}} \Big((1 + \nu) \|\sigma\|^2 - \nu \, \mathrm{tr}(\sigma)^2 \Big) = \frac{1}{2\tilde{Y}} \Big((a + 2b(x_1 + c)x_2)^2 + 2b^2(1 + \nu)(e^2 - x_2^2)^2 \Big).$$

Integrating over the thickness we get

$$\int_\omega \frac{1}{4\tilde{\mu}} \|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})} \mathrm{tr}(\sigma)^2 = \int_{-\ell/2}^{\ell/2} \frac{1}{2\tilde{Y}} \Big(2ea^2 + \frac{8e^3}{3} b^2 (x_1 + c)^2 + \frac{32e^5}{15} (1 + \nu) b^2 \Big) \, dx_1.$$

Direct computations give

$$\int_{-\ell/2}^{\ell/2} \frac{1}{\tilde{Y}(x_1)} \, dx_1 \leq \frac{\ell}{Y} (1 + 2 \frac{k'}{k} \frac{e}{\ell})$$

and

$$\int_{-\ell/2}^{\ell/2} \frac{(x_1 + c)^2}{\tilde{Y}(x_1)} \, dx_1 \leq \frac{\ell^3}{36Y} \left(3 + \left(\frac{6c}{\ell} \right)^2 \right) (1 + 12 \frac{k'}{k} \frac{e}{\ell}).$$

Hence

$$\int_\omega \frac{1}{4\tilde{\mu}} \|\sigma\|^2 - \frac{\tilde{\lambda}}{8\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})} \mathrm{tr}(\sigma)^2 \leq (1 + 15 \frac{k'}{k} \frac{e}{\ell}) \frac{Ye}{\ell} \Big[(2U_1^+)^2 + \frac{e^2}{3} \left(3(\theta^+ + \theta^-)^2 + (\theta^+ - \theta^-)^2 \right) \Big].$$

On the other hand, noticing that the chosen field σ is divergence free and thus that $\int_\omega \sigma : e(u) = \int_{\partial\omega} (\sigma \cdot n) \cdot u$, we obtain

$$\int_\omega \sigma : e(u) \geq (1 - \frac{e}{\ell}) \frac{2Ye}{\ell} \Big[(2U_1^+)^2 + \frac{e^2}{3} \Big(3(\theta^+ + \theta^-)^2 + (\theta^+ - \theta^-)^2 \Big) - \frac{e}{\ell} (v^+ - v^-)^2 \Big].$$

The lemma is proven by collecting these two results.

□

Proof of upperbound lemma 3: By adding if needed a rigid motion to u, we can restrict our attention to the case $U_2^- + \theta^- k'e = U_2^+ - \theta^+ k'e = 0$ and $U_1^- = -U_1^+$. In that case we simply have to state for the energy the upperbound

$$\int_\omega \left(\tilde{\mu} \|e(u)\|^2 + \frac{\tilde{\lambda}}{2} \mathrm{tr}(e(u))^2 \right) \le \frac{Ye}{\ell}\big(1 + C\frac{e}{\ell}\big)\Big[(U_1^+ - U_1^-)^2 + \frac{e^2}{3}\Big(3\gamma^2(\theta^+ + \theta^-)^2 + (\theta^+ - \theta^-)^2\Big)\Big] \tag{40}$$

where $\gamma := \frac{\ell - 2k'e}{\ell} = 1 - 2k'\frac{e}{\ell}$. We introduce the continuous piecewise affine function φ defined by $\varphi(x) = 1$ if $|x| < \frac{\ell}{2} - 2k'e$, $\varphi(x) = 0$ if $|x| > \frac{\ell}{2} - k'e$. Then we define u by setting $u(x_1, x_2) = U^- + \theta^-(-x_2, x_1 + \frac{\ell}{2})$ if $x_1 < -\frac{\ell}{2} + k'e$, $u(x_1, x_2) = U^+ + \theta^+(-x_2, x_1 - \frac{\ell}{2})$ if $x_1 > \frac{\ell}{2} - k'e$ and, for $|x_1| < \frac{\ell}{2} - k'e$,

$$\begin{aligned} u_1(x_1, x_2) &:= (U_1^+ - U_1^-)\frac{x_1}{\gamma \ell} - \frac{1}{4\ell^2}\Big(\frac{12x_1^2}{\gamma^2}(\theta^+ + \theta^-) + \frac{4\ell x_1}{\gamma}(\theta^+ - \theta^-) - \ell^2(\theta^+ + \theta^-)\Big)x_2 \\ u_2(x_1, x_2) &:= \frac{\gamma}{8\ell^2}\Big(\frac{2x_1}{\gamma}(\theta^+ + \theta^-) + \ell(\theta^+ - \theta^-)\Big)\Big(\frac{4x_1^2}{\gamma^2} - \ell^2\Big) \\ &\qquad - \frac{\gamma\nu\varphi(x_1)}{\ell^2}\Big(\ell(U_1^+ - U_1^-)x_2 - \big(\frac{6x_1}{\gamma}(\theta^+ + \theta^-) + \ell(\theta^+ - \theta^-)\big)\frac{x_2^2}{2}\Big). \end{aligned}$$

It is straightforward to check that u belongs to $\mathrm{H}^1(\omega, \mathbb{R}^2)$ and some cumbersome but direct computations lead to estimation (40).

□

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References

[1] H. Abdoul-Anziz, P. Seppecher, effective computation of second gradient stiffnesses of elastic frame lattices, *in preparation.*

[2] E. C. Aifantis, On the role of gradients in the localization of deformation and fracture *International Journal of Engineering Science* **30 (10)** (1992) 1279-1299.

[3] E. C. Aifantis, Strain gradient interpretation of size effects *International Journal of Fracture* **95.1-4** (1999) 299-314.

[4] G. Allaire, Homogenization and two-scale convergence, *SIAM Journal on Mathematical Analysis*, **23 (6)** (1992) 1482-1518.

[5] G. Allaire, M. Briane, M. Vanninathan, A comparison between two-scale asymptotic expansions and Bloch wave expansions for the homogenization of periodic structures *SeMA Journal* **73 (3)** (2016) 237-259.

[6] J.J. Alibert, P. Seppecher, F. dell'Isola, Truss modular beams with deformation energy depending on higher displacement gradients *Mathematics and Mechanics of Solids* **8** (2003) 51-73.

[7] J.J. Alibert, A. Della Corte, Second-gradient continua as homogenized limit of pantographic microstructured plates: a rigorous proof *Zeitschrift für angewandte Mathematik und Physik* **66 (5)** (2015) 2855-2870.

[8] G. Barbagallo, A. Madeo, I. Azehaf, I. Giorgio, F. Morestin, Ph. Boisse, Bias extension test on an unbalanced woven composite reinforcement: Experiments and modeling via a second-gradient continuum approach *Journal of Composite Materials*, (2017) 153-170.

[9] G. Barbagallo, A. Madeo, F. Morestin, Ph. Boisse, Modeling the deep drawing of a 3D woven fabric with a second gradient model, *Mathematics and Mechanics of Solids*, (2016).

[10] I. Babuska, S. A. Sauter, Algebraic algorithms for the analysis of mechanical trusses, *Mathematics of computation* **73, (248)**, (2004) 1601-1622.

[11] G. Bouchitté, M. Bellieud, Homogenization of a soft elastic material reinforced by fibers *Asymptotic Analysis* **32 (2)** (2002) 153-183.

[12] G. Bouchitté, G. Buttazzo, I. Fragala, Mean curvature of a measure and related variational problems, Ann. Scuola Norm. Sup. Pisa, (4), 25 (1997) 179-196.

[13] G. Bouchitté, I. Fragalà, Homogenization of elastic thin structures: a measure-fattening approach *J. Convex Analysis* **9 (2)** (2002) 339–362.

[14] G. Bouchitté, I. Fragalà, Homogenization of Thin Structures by Two-Scale Method with Respect to Measures, *SIAM J. Math. Anal.* **32 (6)** (2006) 1198-1226.

[15] C. Boutin, Microstructural effects in elastic composites *International Journal of Solids and Structures*, Elsevier, **33 (7)** (1996) 1023-1051.

[16] C. Boutin , S. Hans, C. Chesnais, Generalized Beams and Continua, Dynamics of Reticulated Structures *Mechanics of Generalized Continua, Advances in Mechanics and Mathematics* **21** (2010) 131-141.

[17] A. Braides, Γ-convergence for beginners *Oxford University Press* (2002).

[18] A. Braides, M. S. Gelli, Limits of Discrete Systems with Long-Range Interactions*Journal of Convex Analysis* **9 (2)** (2002) 363-399.

[19] M. Briane, M. Camar-Eddine, Homogenization of two-dimensional elasticity problems with very stiff coefficients *Journal de Mathmatiques Pures et Appliques* **88 (6)** 483-505.

[20] P. Casal, La théorie du second gradient et la capillarité *C. R. Acad. Sci. Paris, Série A* **274** (1972) 1571-1574.

[21] M. Camar-Eddine, P. Seppecher, Closure of the set of diffusion functionals with respect to the Mosco-convergence *Mathematical Models and Methods in Applied Sciences* **12 (08)**, (2002) 1153-1176.

[22] M. Camar-Eddine, P. Seppecher, Determination of the closure of the set of elasticity functionals *Arch. Ration. Mech. Anal.***170 (3)** (2003) 211-245.

[23] G. A. Chechkin, V. V. Jikov, D. Lukkassen, A. L. Piatnitski, On homogenization of networks and junction, *Asymptotic Analysis* **30** (2002) 6180.

[24] G.A. Chechkin, A.L. Piatnitski, A.S. Shamaev, Homogenization methods and applications, American Mathematical Society, Providence, RI, USA, 2007.

[25] G. Dal Maso, An introduction to Γ-convergence *Birkhäuser, Boston* (1993).

[26] F. dellIsola, P. Seppecher, Edge Contact Forces and Quasi-Balanced Power, *Meccanica* **32 (1)** (1997) 33-52.

[27] F. dell'Isola, U. Andreaus, L. Placidi, At the origins and in the vanguard of peri-dynamics, non-local and higher gradient continuum mechanics. An underestimated and still topical contribution of Gabrio Piola *Mechanics and Mathematics of Solids* **20 (8)** (2014) 887-928.

[28] A. C. Eringen, *Microcontinuum field theories* Springer-Verlag, New York (2001).

[29] S. Forest, Homogenization methods and the mechanics of generalized continua *Geometry, Continua and Microstructure, ed. by G. Maugin, Travaux en Cours No. 60, Hermann, Paris, France* (1999) 35-48.

[30] G. G. Giusteri, The multiple nature of concentrated interactions in second-gradient dissipative liquids *Zeitschrift für angewandte Mathematik und Physik* **64 (2)** (2013) 371-380.

[31] S. Gonella, M. Ruzzene, Homogenization and equivalent in-plane properties of two-dimensional periodic lattices,*International Journal of Solids and Structures* **45** (2008) 2897-2915

[32] H. Gouin, Utilization of the Second Gradient Theory in Continuum Mechanics to Study the Motion and Thermodynamics of Liquid-Vapor Interfaces *Physicochemical Hydrodynamics, Part E*, NATO ASI Series **174**, Springer US (1988) 667-682.

[33] A. Madeo, M. Ferretti, F. dellIsola, Ph. Boisse, Thick fibrous composite reinforcements behave as special second-gradient materials: three-point bending of 3D interlocks *Zeitschrift für angewandte Mathematik und Physik ZAMP* **66 (4)** (2015) 2041-2060.

[34] P.G. Martinsson and I. Babuska, Mechanics of Materials with Periodic Truss or Frame Micro-structures *Archives for Rational Mechanics and Analysis* **185 (2)** (2007) 201-234.

[35] P.G. Martinsson and I. Babuska, Homogenization of materials with with periodic truss or frame micro-structures *Mathematical Models and Methods in Applied Sciences* **17 (5)** (2007) 805-832.

[36] M. Mazière and S. Forest, Strain gradient plasticity modeling and finite element simulation of Lüders band formation and propagation *Continuum Mechanics and Thermodynamics,* **27** (2015) 83-104.

[37] N. Meunier, 0. Pantz, A. Raoult, Elastic limit of square lattices with three-point interactions, *Mathematical Models and methods in Applied Sciences* **22 (11)** (2012) 1250032.

[38] R. D. Mindlin, Influence of Couple-stresses on Stress Concentrations *Experimental Mechanics* **3 (6)** (1962) 756-757.

[39] R. D. Mindlin, H. F. Tiersten, Effects of couple-stresses in linear elasticity *Arch. Rational Mech. Anal.* **11** (1962) 415-448.

[40] R. D. Mindlin, Second gradient of strain and surface tension in linear elasticity *Int. J. Solids and Struct* **1 (4)** (1965) 417-438.

[41] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989), 608-623.

[42] M. Ostoja-Starzewski, Lattice models in micromechanics, *Appl Mech Rev* **55, n(1)**, (2002) 35-59.

[43] S. E. Pastukhova, Homogenization of problems of elasticity theory on periodic box and rod frames of critical thickness, *Journal of Mathematical Sciences* **130 (5)** (2005) 4954-5004.

[44] C. Pideri, P. Seppecher, A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium *Continuum Mechanics and Thermodynamics* **9 (5)** (1997) 241-257.

[45] C. Polizzotto, G. Borino, A thermodynamics-based formulation of gradient-dependent plasticity *European Journal of Mechanics-A/Solids* **17 (5)** (1998) 741-761.

[46] A. Raoult, D. Caillerie, A. Mourad, Elastic lattices: equilibrium, invariant laws and homogenization *Annali dell Universita di Ferrara* **54 (2)** (2008) 297-318.

[47] G. Sciarra, F. dell'Isola, O. Coussy, Second gradient poromechanics *International Journal of Solids and Structures* **44 (20)** (2007) 6607-6629.

[48] P. Seppecher, Moving contact lines in the Cahn-Hilliard theory *International journal of engineering science* **34 (9)** (1996) 977-992.

[49] P. Seppecher, J.J. Alibert, F. dell'Isola, Linear elastic trusses leading to continua with exotic mechanical interactions*Journal of Physics: Conference Series* **319 (1)**, (2011) 012018.

[50] V. P. Smyshlyaev, K. D. Cherednichenko, On rigorous derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media, *J. Mech. Phys. Solids* **48** (2000) 1325-1357.

[51] R. A. Toupin, Elastic Materials with couple-stresses. *Arch. Rat. Mech. Analysis* **11** (1962) 385-414.

[52] R. A. Toupin, Theories of elasticity with couple-stress *Arch. Rational Mech. Anal.* **17** (1964) 85-112.

[53] N. Triantafyllidis, E. C. Aifantis, A gradient approach to localization of deformation. I. Hyperelastic materials *Journal of Elasticity* **16 (3)** (1986) 225-237.

[54] Y. Yang, A. Misra, Micromechanics based second gradient continuum theory for shear band modeling in cohesive granular materials following damage elasticity *International Journal of Solids and Structures* **49 (18)** (2012) 2500-2514.