Observability: Range-Only Versus Bearings-Only Target Motion Analysis When the Observer Maneuvers Smoothly

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***Abstract*— Range-only target motion analysis (ROTMA) consists of estimating the trajectory of a target using a single platform collecting range-only measurements. Observability analysis is carried out when the target is in constant-velocity motion and the observer maneuvers gently (a constant turn motion or a constant acceleration motion). We compare observability in bearings-only target motion analysis (BOTMA) and ROTMA throughout the paper, together with the rank of the Fisher information matrix. In each case, we establish necessary and sufficient observability conditions and we identify the virtual (or ghost) targets giving the same measurements when the system is not observable.**

***Index Terms*— Target motion analysis, tracking, range-only, bearings-only, observability, Fisher information matrix, constant turn motion, constant acceleration motion.**

I. Introduction

THIS PAPER presents the second part of the observability analysis in range-only target motion analysis (ROTMA), started in a previous paper [13]. The target (or source) was assumed to be in constant velocity (CV) motion. The previous paper [13] provided the following results:

The Fisher information matrices (FIMs) in BOTMA and in ROTMA have the same rank when the source is in CV motion, whatever the trajectory of the observer.

Secondly, when the observer is zigzagging, with constant speed on each leg, the trajectory of the source is observable in ROTMA if and only if it is unobservable in BOTMA. We proved this happens when and only when the bearings are constant. In this case, the FIM is singular, whereas in ROTMA, the target's trajectory is observable. Conversely, when the trajectory of the target is observable in BOTMA, it is unobservable in ROTMA.

At this point, a prior question must be asked: are these antinomic observability conditions in BOTMA and in ROTMA maintained for other observers' maneuvers? Some answer elements are given in the present paper for two types of smooth maneuvers of the observer: a constant turn (CT) motion or a constant acceleration (CA) motion and a combination, CA-CV (for BOTMA only) and CT-CV.

Note that generalizing this comparison is difficult because, unlike in BOTMA, general necessary and sufficient observability conditions do not exist in ROTMA: indeed, when the observer maneuvers, this problem cannot be expressed by a linear system, whereas in BOTMA, a linear system can be exhibited [3, 11, 16]. In ROTMA, for each observer's kinematics, a specific study of observability must be conducted.

Our strategy of analysis is the same as in [13]: we start with BOTMA, we deduce the rank of the FIM, and finally, we study the observability of the target in ROTMA (under the same conditions).

We will prove that when the observer is in CT motion, the target is observable in BOTMA and in ROTMA as well. In this case, the FIMs are of full rank (no antinomy). When the observer is in CA motion, several situations can occur: the target can be observable in ROTMA and the FIM can be of deficient rank, or the target can be unobservable in ROTMA and the FIM can be of full-rank or deficient rank.

When the trajectory of the target of interest is not observable, a set of ghost-targets exists, which misleads the operator when he or she tries to estimate the state vector. This problem also occurs when associating multi-tracks between several arrays [6]. Therefore, in BOTMA, the observer must maneuver in order to render the trajectory of the target observable. Unfortunately, some maneuvers are ineffective [7] [12]. Here, we give some examples of ineffective maneuvers in ROTMA.

The paper is organized as follows:

In Section II, the problem and the ad-hoc notations are given. The notion of observability is recalled and the models of the two types of trajectories of the observer are given, as well. We introduce the notion of a rendezvous route (and the associated criteria), which will play a crucial role in the analysis of observability when the observer is in CA motion.

In Section III, observability in BOTMA is analyzed when the observer is in CT motion, in CA motion, and in CA motion followed by CV motion or the converse. For each kinematic, we give necessary and sufficient observability conditions. The set of ghost-targets is characterized in the scenarios where the target is unobservable.

Section IV is devoted to ROTMA: we start by revisiting the situations in which the observer is in CT motion and then in CA motion. For each of them, we give necessary and sufficient observability conditions. In unobservable cases, we identify the set of ghost-targets. Illustrative examples are

C. Jauffret and A.C. Pérez are with Université de Toulon, CNRS, IM2NP (UMR 7734), CS 60584, 83041 Toulon Cedex 9, France (email: annie-claude.perez, jauffret@univ-tln.fr)
D. Pillon, is retired and works as counselor with IM2NP (email: pillon_denis@orange.fr)

given.
The conclusion follows.

II. General notion and notations

A. Definitions and notations

A target (T) and an observer (O) move in the same plane, given a Cartesian system. The target has a CV all along the scenario, while the observer maneuvers (here, the term "maneuver" is employed when the observer is not in CV motion). The scenario starts at time $t=0$ and finishes at time $t=T_f$.

For the observer, the position and velocity at time t are respectively $P_O(t)=[x_O(t)\ y_O(t)]^\intercal$ and $V_O(t)=\dfrac{dP_O(t)}{dt}=[\dot{x}_O(t)\ \dot{y}_O(t)]^\intercal$. Both are concatenated into the vector $X_O(t)=\left[x_O(t)\quad y_O(t)\quad \dot{x}_O(t)\quad \dot{y}_O(t)\right]^\intercal$.

For the target, the notations are similar: $P_T(t)=[x_T(t)\ y_T(t)]^\intercal$, $V_T=\dfrac{dP_T(t)}{dt}=[\dot{x}_T\ \dot{y}_T]^\intercal$, and $X_T(t)=\left[x_T(t)\quad y_T(t)\quad \dot{x}_T\quad \dot{y}_T\right]^\intercal$.

In the sequel, $X_T(0)$ will be simply denoted as X_T, which entirely defines the target's trajectory. Obviously, $P_T(t)=P_T(0)+tV_T$.

We will assume that $P_T(t)\neq P_O(t),\ \forall t\in\left[0,\ T_f\right]$.

The motion of the target relative to the observer is given by $P_{OT}(t)=P_T(t)-P_O(t)=[x_{OT}(t)\ y_{OT}(t)]^\intercal$ and by $V_{OT}(t)=\dfrac{dP_{OT}(t)}{dt}=[\dot{x}_{OT}(t)\ \dot{y}_{OT}(t)]^\intercal$. The relative velocity vector. We define the vector $X_{OT}(t)=X_T(t)-X_O(t)=\left[x_{OT}(t)\quad y_{OT}(t)\quad \dot{x}_{OT}(t)\quad \dot{y}_{OT}(t)\right]^\intercal$.

All the angles are clockwise-positive. Subsequently, we will use the symbol $\angle$ to designate angles: for any pair of vectors U and W, $\angle(\mathsf{U},\mathsf{W})$ is the angle defined by the couple (U,W) referenced to U. When U is collinear to the northward direction, we will use $\angle\mathsf{W}$ only (for the bearing or heading).

The range and the bearing at time t are given respectively by $r(t)=\|P_{OT}(t)\|$ and $\theta(t)=\angle P_{OT}(t)$. So, $P_{OT}(t)=r(t)\begin{bmatrix}\sin\theta(t)\\ \cos\theta(t)\end{bmatrix}$; that is, $\begin{cases}x_T(t)=r(t)\sin\theta(t)+x_O(t)\\ y_T(t)=r(t)\cos\theta(t)+y_O(t)\end{cases}$.

Subsequently, we will use the following simplified notations: $\theta_0=\theta(0)$, $r_0=r(0)$, $v_r=\|V_{OT}(0)\|$, and $h_r=\angle V_{OT}(0)$ for the initial bearing, range, relative speed, and heading. For convenience, $X_{OT}(0)$ will be denoted simply X_{OT}.

Figure 1 displays a typical scenario.

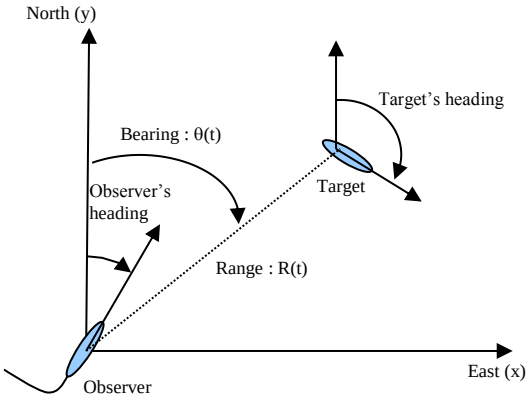

Fig. 1. Typical scenario of TMA.

B. Observability notion

We extend the previous notation to emphasize this dependence: $\theta(t)$ and $r(t)$ can be denoted $\theta(t,\ X_{OT})$ and $r(t,\ X_{OT})$.

We recall that the target's trajectory is declared observable in BOTMA if the following statement is true: $\forall t\in\left[0,\ T_f\right],\ \theta(t,\ X)=\theta(t,\ X_{OT})\ \Rightarrow\ X=X_{OT}$. Otherwise, the trajectory is said to be unobservable: at least, one vector $X_{OG}=\left[x_{OG}(0)\quad y_{OG}(0)\quad \dot{x}_{OG}\quad \dot{y}_{OG}\right]^\intercal$ (defining a CV motion) different from X_{OT} exists such that $\theta(t,\ X_{OG})=\theta(t,\ X_{OT}),\ \forall t\in\left[0,\ T_f\right]$.

Similarly, in ROTMA, the observability definition is $\forall t\in\left[0,\ T_f\right],\ r(t,\ X)=r(t,\ X_{OT})\ \Rightarrow\ X=X_{OT}$. Otherwise, again, at least one vector $X_{OG}=\left[x_{OG}(0)\quad y_{OG}(0)\quad \dot{x}_{OG}\quad \dot{y}_{OG}\right]^\intercal$ different from X_{OT} exists such that $r(t,\ X_{OG})=r(t,\ X_{OT})$.

The vector $X_G=X_{OG}+X_O$ defines the "virtual" trajectory of a "ghost-target", denoted G. We define similarly $P_G(0)=\left[x_G(0)\quad y_G(0)\right]^\intercal$, $V_G=[\dot{x}_G\ \dot{y}_G]^\intercal$, $P_G(t)=P_G(0)+tV_G=[x_G(t)\ y_G(t)]^\intercal$, and $X_G(t)=\left[x_G(t)\quad y_G(t)\quad \dot{x}_G\quad \dot{y}_G\right]^\intercal$, with the convention $X_G=X_G(0)$.

Note that, depending on various circumstances, the set of "ghosts" is finite or can be a family (an uncountable set).

Observability analysis has two aims:

a) Give a necessary and sufficient condition to have unicity of the state vector X_{OT}.
b) When this condition is not satisfied, characterize the set of X_{OG}.

This will conduct our paper.

C. Observer kinematic models

As announced in the introduction, two models of smooth observer motion are considered in this paper: in the first, the observer travels in an arc of a circle at constant speed, and in the second, it has a constant acceleration vector.

1) CT motion

The observer turns around a fixed point $P_C = \begin{bmatrix} x_C \\ y_C \end{bmatrix}$ at range $\rho > 0$, with a constant turn rate $\omega \neq 0$ (positive if the motion of the observer is clockwise) and an "initial angle" φ relative to north, at the beginning of its motion. Its speed is constant. As a consequence, at time t (recall that the initial time is equal to 0), the location of the observer is given by $P_O(t) = P_C + \rho \begin{bmatrix} \sin(\omega t + \varphi) \\ \cos(\omega t + \varphi) \end{bmatrix}$ (see Fig. 2). In order to simplify the coming calculation, we will assume that $P_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

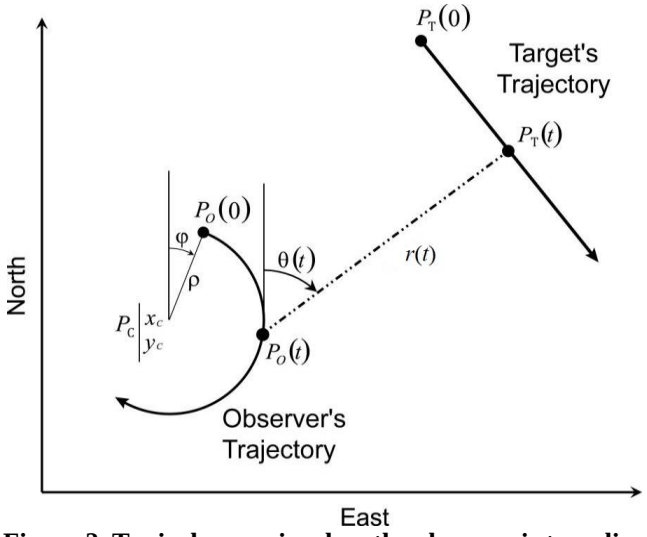

Figure 2. Typical scenario when the observer is traveling in an arc of a circle.

2) CA motion

The position of the observer at any time t is $P_O(t) = P_O(0) + tV_O(0) + \frac{t^2}{2}\Gamma$, where $V_O(0)$ is the initial velocity and $\Gamma = \begin{bmatrix} \gamma_x & \gamma_y \end{bmatrix}^T$ is the (non-zero) acceleration vector. The relative position of the target with respect to the observer is

$$P_{OT}(t) = P_{OT}(0) + tV_{OT}(0) - \frac{t^2}{2}\Gamma \quad (1)$$

Without loss of generality, we will assume that $\gamma_x < 0$ and $\gamma_y = 0$. Indeed, a suitable rotation of the entire scenario allows us to be in this case.[1] This assumption will make easier the following observability analysis.

[1] The matrix of this rotation is $\begin{bmatrix} -\sin\angle\Gamma & -\cos\angle\Gamma \\ \cos\angle\Gamma & -\sin\angle\Gamma \end{bmatrix}$.

D. Rendezvous routes in CA motion

The observability criteria when the observer is in CA motion will be shown to be linked to the rendezvous (or collision) route in Sections III. C. (BOTMA) and IV. B. 2. (ROTMA).

Definition: the rendezvous route

The target and the observer are said to be on a rendezvous route (RDVR), when they are in the same place at a time t_c.

Actually, this rendezvous instant is purely virtual: before $t = 0$ and after T_f, O and T were and will be free to choose their own trajectories. Note that this motion model is not the pursuit curve motion, which has not been studied in the TMA observability problem, from our knowledge.

1) The two types of RDVR

Proposition 1: General properties of RDVR

If O and T are on an RDVR, then

- *Either* $P_{OT}(0)$, $V_{OT}(0)$ *and* Γ *are collinear,*
- *Or* $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear.*

Proof:

O and T collide at t_c if and only if

$$\begin{cases} x_{OT}(0) + t_c \dot{x}_{OT}(0) - \frac{1}{2} t_c^2 \gamma_x = 0 & (2) \\ y_{OT}(0) + t_c \dot{y}_{OT}(0) = 0 & (3) \end{cases}$$

If $y_{OT}(0) = 0$, then (3) implies that $\dot{y}_{OT}(0) = 0$. The vectors $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear.

Else, $\dot{y}_{OT}(0) = -\frac{y_{OT}(0)}{t_c} \neq 0$. In other words, $P_{OT}(0)$ and Γ are noncollinear, and $V_{OT}(0)$ and Γ are noncollinear.

QED.

Definition: The two types of RDVR

The RDVRs are called rendezvous routes of type I (RDVR-I), when $P_{OT}(0)$, $V_{OT}(0)$, *and* Γ *are collinear. When* $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear as well, the RDVRs are called rendezvous routes of type II (RDVR-II).*

Note that for the RDVR-II, $P_{OT}(0)$ *and* $V_{OT}(0)$ can be collinear. Fig. 3 and Fig. 4 depict an example of RDVR-I, and of RDVD-II, respectively. We can notice that the bearings are constant for the RDVR-I whereas they change in time for the RDVR-II. We will see why and how subsequently.

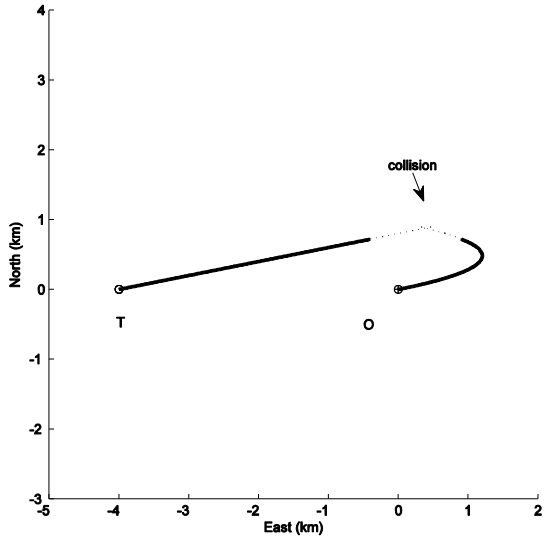

Figure 3. Example of RDVR-I.

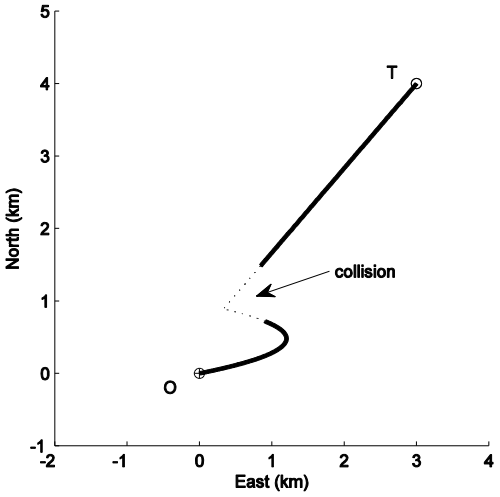

Figure 4. Example of RDVR-II.

The converse of Proposition 1 is given in the following two propositions.

Proposition 2: Condition of RDVR-I

Assume that $P_{OT}(0)$, $V_{OT}(0)$ *and* Γ *are collinear; that is,* $P_{OT}(0)=\eta\Gamma$ *(with* $\eta\neq 0$*) and* $V_{OT}(0)=\lambda\Gamma$. *O and T are on an RDVR if and only if* $\lambda^2 \geq -2\eta$.

Proof:

O and T are on an RDVR, if and only if $x_{OT}(0)+t_c\dot{x}_{OT}(0)-\frac{1}{2}t_c^2\gamma_x=0$ (we do not have any equation with the y-component since $\gamma_y=0$). This equality is equivalent to

$$\left(\eta+\lambda t_c-\frac{1}{2}t_c^2\right)\gamma_x=0 \text{ or } t_c^2-2\lambda t_c-2\eta=0.$$

Hence, O and T are on an RDVR if and only if the equation $t^2-2\lambda t-2\eta=0$ has one or two real roots (one of them is t_c), that is, if and only if the discriminant $\Delta=\lambda^2+2\eta$ is positive.

QED.

Obviously, the assumption "$P_{OT}(0)$, $V_{OT}(0)$ *and* Γ *are collinear* " is not sufficient to characterize an RDVR-I, but implies that the bearings are constant. Conversely, if the bearings are constant, the target and the observer are not necessarily on an RDVR-I.

Let us note that in this case, the bearings are piecewise equal to $\pm\frac{\pi}{2}$, and Proposition 2 remains valid up to a rotation, that is, for $\theta(t)=\text{constant}$ up to t_c. Note also that two rendezvous instants may exist (depending on Δ). In Fig. 5, an example of such a situation is given: The target starts at $[-4000 \quad 0]^T$ (m) with a velocity of $[20 \quad 2]^T$ (m/s); the observer starts at $[0 \quad 0]^T$ (m) with a velocity of $[10 \quad 2]^T$ (m/s) and its acceleration vector is $[-0.0416 \quad 0]^T \quad (\text{m/s}^2)$.

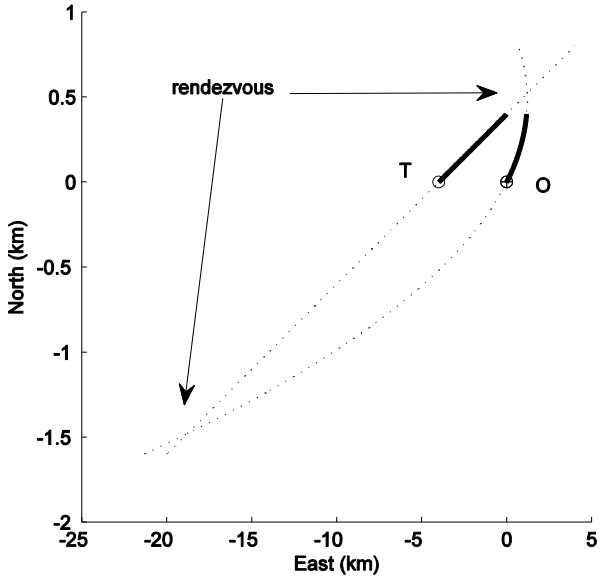

Figure 5. Example of RDVR-I where the target and the observer have two RDVs.

Proposition 3: Condition of RDVR-II.

Assume that $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear as well.*

O and T are on an RDVR if and only if $\gamma_x=2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$.

Proof:

Equation (3) implies that $t_c=-\frac{y_{OT}(0)}{\dot{y}_{OT}(0)}$. Substituting into (2),

we get $x_{OT}(0)-\frac{y_{OT}(0)}{\dot{y}_{OT}(0)}\dot{x}_{OT}(0)-\frac{1}{2}\frac{y_{OT}^2(0)}{\dot{y}_{OT}^2(0)}\gamma_x=0$. We end up with $\gamma_x=2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$

QED.

Note that, an unique RDV instant exists for an RDVD-II.
In the two following propositions, we give a criterion based on the measurements (range or bearing), which allows us to know which type of RDVR we are on.

2) Criterion of RDVR-I

From the remark following Proposition 2, when the target and the observer are on an RDVR-I, the bearings are constant. In this case, the trajectory of the target is not observable in BOTMA. In ROTMA, we will see that this type of route must be considered in the observability analysis (see Proposition 15).
The following proposition gives us a criterion of RDVR-I based only on range. In practice, when the measurements are corrupted by additive noises, it can help to construct a statistical test to decide if the two mobiles are on an RDVR-I.

Proposition 4: Criterion on range of RDVR-I
The target and the observer are on RDVR-I if and only if a scalar $\dot{r}$ *exists such that, for* $t\in[0,T_f]$, either $r(t)-\frac{1}{2}t^2\gamma_x=r_0+t\dot{r}$, *or* $r(t)+\frac{1}{2}t^2\gamma_x=r_0+t\dot{r}$ *and* $\dot{r}^2\geq -2r_0\gamma_x$.

Proof:
First, assume that the target and the observer are on an RDVR-I.

From (1), we have $x_{OT}(t)=x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{t^2}{2}\gamma_x=\left(\eta+t\lambda-\frac{1}{2}t^2\right)\gamma_x$, and $r(t)=|x_{OT}(t)|$. We denote by t_1 and t_2 (with $t_1\leq t_2$) the roots of the equation $x_{OT}(t)=0$. Note that they can be the same.
If $[0,T_f]\subset[t_1,t_2]$, then $r(t)=-x_{OT}(t)$. If $T_f<t_1$, or $t_2<0$ then $r(t)=x_{OT}(t)$. So, we get the result.

Conversely, suppose now that either $r(t)-\frac{1}{2}t^2\gamma_x=r_0+t\dot{r}$, or $r(t)+\frac{1}{2}t^2\gamma_x=r_0+t\dot{r}$ for $t\in[0,T_f]$ and $\dot{r}^2\geq -2r_0\gamma_x$. Let us prove that the observer and the target are on an RDVR. To avoid repetition, we will only consider the case $r(t)+\frac{1}{2}t^2\gamma_x=r_0+t\dot{r}$ and $\dot{r}^2\geq -2r_0\gamma_x$.
We have to solve the following equation: $\left\|P_{OT}(0)+tV_{OT}(0)-\frac{1}{2}t^2\Gamma\right\|^2=\left(r_0+t\dot{r}-\frac{1}{2}t^2\gamma_x\right)^2$.

After developing the left and right expressions and equating the coefficients of each power of t, we end up with a system of four equations:

$\|P_{OT}(0)\|^2=r_0^2$ (4)
$P_{OT}^T(0)V_{OT}(0)=r_0\dot{r}$ (5)
$\|V_{OT}(0)\|^2-P_{OT}^T(0)\Gamma=\dot{r}^2-r_0\gamma_x$ (6)
$V_{OT}^T(0)\Gamma=\dot{r}\gamma_x$ (7)

In terms of components, they are equivalent to
$x_{OT}^2(0)+y_{OT}^2(0)=r_0^2$ (8)
$x_{OT}(0)\dot{x}_{OT}(0)+y_{OT}(0)\dot{y}_{OT}(0)=r_0\dot{r}$ (9)
$\dot{x}_{OT}^2(0)+\dot{y}_{OT}^2(0)-x_{OT}(0)\gamma_x=\dot{r}^2-r_0\gamma_x$ (10)
$\dot{x}_{OT}(0)\gamma_x=\dot{r}\gamma_x$ (11)

From (11), we get $\dot{x}_{OT}(0)=\dot{r}$. Substituting into (9) and (10), we obtain
$y_{OT}(0)\dot{y}_{OT}(0)=-(x_{OT}(0)-r_0)\dot{r}$ (12)
$\dot{y}_{OT}^2(0)=(x_{OT}(0)-r_0)\gamma_x$ (13)

Now, from (8), we have
$y_{OT}^2(0)=-(x_{OT}(0)-r_0)(x_{OT}(0)+r_0)$ (14)

Taking the square of each side of (12), after replacing $\dot{y}_{OT}^2(0)$ and $y_{OT}^2(0)$ with their respective expressions given by (13) and (14), we get
$-(x_{OT}(0)-r_0)^2(x_{OT}(0)+r_0)\gamma_x=(x_{OT}(0)-r_0)^2\dot{r}^2$. (15).

Let us examine two cases:

Case 1 : $x_{OT}(0)\neq r_0$

Equation (15) is equivalent to $x_{OT}(0)=-\left(r_0+\frac{\dot{r}^2}{\gamma_x}\right)$.
Substituting into (14), we finally get $y_{OT}^2(0)=-\left(2r_0+\frac{\dot{r}^2}{\gamma_x}\right)\frac{\dot{r}^2}{\gamma_x}$.

Consequently, $2r_0+\frac{\dot{r}^2}{\gamma_x}\geq 0$; that is, $2r_0\gamma_x+\dot{r}^2\leq 0$.
But, by assumption, we have $2r_0\gamma_x+\dot{r}^2\geq 0$. Hence $2r_0\gamma_x+\dot{r}^2=0$. (16)

It follows that $y_{OT}(0)=0$; hence $x_{OT}^2(0)=r_0^2$. We conclude that $x_{OT}(0)=-r_0$. Reporting this value into (12), we get $\dot{r}=0$. From (16), we have $r_0=0$, which is incompatible with the assumption $P_{OT}(t)\neq 0$ for $t\in[0,T_f]$. This must be discarded.

Case 2 : $x_{OT}(0)=r_0$

From (13) and (14), $y_{OT}(0)=0$ and $\dot{y}_{OT}(0)=0$; that is $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear. With the inequality $\dot{r}^2 \geq -2r_0\gamma_x$ being equivalent to $\lambda^2 \geq -2\eta$, the target and the observer are on an RDVR-I, from Proposition 2.

QED.

3) Criterion of RDVR-II

Similarly to ROTMA, BOTMA observability conditions (see Proposition 9) need to have a criterion of RDVR-II.

Proposition 5: Criterion on bearing of RDVR-II
O and T are on RDVR-II if and only if $\tan\theta(t)=\mu_0+\mu_1 t$, *with* $\mu_1 \neq 0$.

The respective values of μ_0 and μ_1 *are* $\frac{x_{OT}(0)}{y_{OT}(0)}$, *and* $\frac{y_{OT}(0)\dot{x}_{OT}(0)-\dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$.

Proof:
We have to prove the equivalence

$$\tan\theta(t)=\mu_0+\mu_1 t \;\Leftrightarrow \gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$$

We have $\tan\theta(t)=\dfrac{x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{1}{2}t^2\gamma_x}{y_{OT}(0)+t\,\dot{y}_{OT}(0)}$.

If $\tan\theta(t)=\mu_0+\mu_1 t$ then

$$x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{1}{2}t^2\gamma_x=\left[y_{OT}(0)+t\,\dot{y}_{OT}(0)\right]\left[\mu_0+\mu_1 t\right].$$

As a consequence, $t_c=-\dfrac{y_{OT}(0)}{\dot{y}_{OT}(0)}$ is a root of $x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{1}{2}t^2\gamma_x$. We get the equality $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$.

Conversely, if $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$, then we readily verify that $t_c=-\dfrac{y_{OT}(0)}{\dot{y}_{OT}(0)}$ is a root of $x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{1}{2}t^2\gamma_x$. Hence, there are two real numbers μ_0 and μ_1 such that

$$x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{1}{2}t^2\gamma_x=\left[y_{OT}(0)+t\,\dot{y}_{OT}(0)\right]\left[\mu_0+\mu_1 t\right].$$

QED.

To resume,
the two mobiles are on an RDVR-II $\Leftrightarrow \tan\theta(t)=\mu_0+\mu_1 t$, and $\mu_1 \neq 0$,
the two mobiles are on an RDVR-I $\Rightarrow \tan\theta(t)=\mu_0$.

It is well known that when the observer is itself in CV motion and the bearings are constant, the observer will meet eventually the target (the collision instant can equal to infinity if the two routes are parallel. To avoid this collision hazard well known by sailors, the observer must maneuver. However, if it accelerates and if the bearing tangent is linear, it may still remain on a rendezvous route.

III. OBSERVABILITY CRITERIA IN BOTMA

Let us recall that in BOTMA, the target trajectory is observable if and only if the FIM is nonsingular. We give hereafter two general results, necessary to prove observability when the trajectory of the target is a combination of CA and CV motions (see proposition 11).

A. Two general results

Proposition 6: Observability equivalence between two observers in BOTMA
Let there be two observers measuring the same bearings. If the target is observable from one, it will be observable from the other (or equivalently, if it is unobservable from one, it will be unobservable from the other).

Proof:
We recall that BOTMA has a linear version, whatever the trajectory of the ownship. Indeed, the noise-free measurement equation $\theta(t)=\tan^{-1}\left[\dfrac{x_T(0)+t\,\dot{x}_T-x_O(t)}{y_T(0)+t\,\dot{y}_T-y_O(t)}\right]$ can be transformed into the linear equation

$$\left[\cos\theta(t) \;\; -\sin\theta(t) \;\; t\cos\theta(t) \;\; -t\sin\theta(t)\right]X_T$$
$$= x_O(t)\cos\theta(t)-y_O(t)\sin\theta(t),\ \forall\, t\in\{t_1,\cdots,t_N\}$$

or, in short, $\mathbf{A}(\theta)X_T=Z$, where the k-th line of $\mathbf{A}(\theta)$ is $\left[\cos\theta(t_k) \;\; -\sin\theta(t_k) \;\; t_k\cos\theta(t_k) \;\; -t_k\sin\theta(t_k)\right]$, and the k-th element of Z is $x_O(t_k)\cos\theta(t_k)-y_O(t_k)\sin\theta(t_k)$; see [3, 11, 16]. The observability is hence brought by the set $\{\theta(t_1),\cdots,\theta(t_N)\}$, which means that if two observers collect the same set $\{\theta(t_1),\cdots,\theta(t_N)\}$, if the target is observable from one, it will be observable from the second one. In this case, the state vector is computed by $X_T=\left[\mathbf{A}^{\mathrm{T}}(\theta)\,\mathbf{A}(\theta)\right]^{-1}Z$.

QED.

Proposition 7: Observability equivalence for time-reversed bearings
Let there be two observers #1 and #2. Observer #2 measures the same bearings as observer #1, but in the inverse temporal order, that is $\theta_{(2)}(t_k)=\theta_{(1)}(t_N-t_k)$, *where* $\theta_{(i)}(t_k)$ *is the bearing measured at time* t_k *by observer #i. If the target detected by*

observer #1 is observable, then the target detected by observer #2 will be, and the converse.

Proof:
Let us define the matrix $\mathbf{A}(\theta_{(i)})$ whose k-th line is $\left[\cos\theta_{(i)}(t_k) \quad -\sin\theta_{(i)}(t_k) \quad t_k\cos\theta_{(i)}(t_k) \quad -t_k\sin\theta_{(i)}(t_k)\right]$. Let us prove that $\text{Rank}\{\mathbf{A}(\theta_{(1)})\} = \text{Rank}\{\mathbf{A}(\theta_{(2)})\}$. We note that the k-th line of $\mathbf{A}(\theta_{(2)})$ is

$$\left[\cos\theta_{(2)}(t_k) \quad -\sin\theta_{(2)}(t_k) \quad t_k\cos\theta_{(2)}(t_k) \quad -t_k\sin\theta_{(2)}(t_k)\right]$$
$$= \left[\cos\theta_{(1)}(t_N - t_k) \quad -\sin\theta_{(1)}(t_N - t_k) \quad t_k\cos\theta_{(1)}(t_N - t_k) \quad -t_k\sin\theta_{(1)}(t_N - t_k)\right]$$

We construct a third matrix denoted $\tilde{\mathbf{A}}$ by permutation of the lines of $\mathbf{A}(\theta_{(2)})$, that is, the first line of $\tilde{\mathbf{A}}$ is the N-th line of $\mathbf{A}(\theta_{(2)})$, the second line of $\mathbf{A}(\theta_{(2)})$ is the (N – 1)-th of $\mathbf{A}(\theta_{(2)})$, and so on. In other words, we flip the matrix in the up/down direction. Obviously, $\text{Rank}\{\tilde{\mathbf{A}}\} = \text{Rank}\{\mathbf{A}(\theta_{(2)})\}$. We note that the first two columns of $\tilde{\mathbf{A}}$ are the first two columns of $\mathbf{A}(\theta_{(1)})$. The third (resp. fourth) column of $\tilde{\mathbf{A}}$ is t_N multiplied by the first (resp., second) column of t_N, minus the third (resp., fourth) column of $\mathbf{A}(\theta_{(1)})$. Hence, $\text{Rank}\{\tilde{\mathbf{A}}\} = \text{Rank}\{\mathbf{A}(\theta_{(1)})\}$. Consequently, $\text{Rank}\{\mathbf{A}(\theta_{(2)})\} = \text{Rank}\{\mathbf{A}(\theta_{(1)})\}$. We readily deduce that $\text{Rank}\{\mathbf{A}^{\mathsf{T}}(\theta_{(1)})\,\mathbf{A}(\theta_{(1)})\} = \text{Rank}\{\mathbf{A}^{\mathsf{T}}(\theta_{(2)})\,\mathbf{A}(\theta_{(2)})\}$.

QED.

Note that these properties cannot be extended for any measurements such as frequency measurements because the Doppler effect is not time-reversible.

B. *Observer in CT motion*

The CT motion was defined in Section II.C.1. We propose the following result when the noise-free bearings are continuously available during $[0, T_f]$.

Proposition 8: Observability in BOTMA for CT motion
If the observer is traveling along an arc of a circle, then any target moving with a constant velocity is observable in BOTMA.

Proof:
Suppose that a ghost (G) moving in CV motion is detected in the same bearings as the target for any $t \in [0, T_f]$.

The equality $\theta(t) = \angle P_{OT}(t) = \angle P_{OG}(t)$ is equivalent to

$P_{OG}(t) = k(t)P_{OT}(t)$ for certain $k(t) > 0$
$\Leftrightarrow P_G(t) - P_O(t) = k(t)[P_T(t) - P_O(t)], \quad \forall\, t \in [0, T_f]$
$\Leftrightarrow P_G(t) = k(t)P_T(t) + [k(t) - 1]P_O(t)$
$$\Leftrightarrow P_G(0) + tV_G = k(t)[P_T(0) + tV_T] + \rho[k(t) - 1]\begin{bmatrix}\sin(\omega t + \varphi)\\ \cos(\omega t + \varphi)\end{bmatrix}$$
$\Leftrightarrow k(t) = 1$.

In other words, no such ghost exists: the trajectory of the target is hence observable.

QED.

Note that if the observer's trajectory contains at least one arc of a circle, then the trajectory of any target having a constant velocity is observable in BOTMA.

C. *Observer in CA motion*

Proposition 9: Observability criterion in BOTMA for CA motion
Assume that the observer is in CA motion.
The target's trajectory is observable in BOTMA if and only if O and T are not on an RDVR and the bearings are not constant.

Proposition 10: Set of ghosts in BOTMA for an RDVR-II
Assume that the observer is in CA motion.
The target's trajectory is unobservable if and only if $\tan\theta(t) = \mu_0 + \mu_1 t$.
If $\mu_1 \neq 0$, *the target and the observer are on a RDVR-II and the ghosts are on a RDVR-II with the observer. Their trajectories are defined by* $X_G(0) = X_T(0) + \alpha\Xi$, *where* α *is a scalar and* Ξ *is a vector in the null space of the matrix* $\begin{bmatrix} y_{OT}(0) & -x_{OT}(0) & 0 & 0 \\ \dot{y}_{OT}(0) & -\dot{x}_{OT}(0) & y_{OT}(0) & 0 \\ 0 & -\frac{\gamma_x}{2} & -\dot{y}_{OT}(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The values of μ_0 *and* μ_1 *are given in Proposition 5.*
If $\mu_1 = 0$, the trajectories of the ghosts are defined by $X_G(0) = X_T(0) + \Xi'$, *with* $\Xi' = [\xi_1' \quad 0 \quad \xi_2' \quad 0]^{\mathsf{T}}$, *for any* ξ_1' *and* ξ_2' *such that* $\xi_1'\xi_2' \neq 0$.

Proof of the two previous propositions:

We have to solve the equation $\theta(t, X) = \theta(t, X_{OT})$. Since the implication
$\theta(t, X) = \theta(t, X_{OT}) \Rightarrow \tan\theta(t, X) = \tan\theta(t, X_{OT})$ holds, we concentrate our effort on the equation $\tan\theta(t, X) = \tan\theta(t, X_{OT})$. We define the components of X by $[x \quad y \quad \dot{x} \quad \dot{y}]^{\mathsf{T}}$.

Two cases must be studied:

Case (1): $y_{OT}(0) \neq 0$ or $\dot{y}_{OT}(0) \neq 0$.
$\tan\theta(t, X) = \tan\theta(t, X_{OT}) \Rightarrow$

$$\frac{x+t\dot{x}-\frac{t^2}{2}\gamma_x}{y+t\dot{y}}=\frac{x_{OT}(0)+t\dot{x}_{OT}(0)-\frac{t^2}{2}\gamma_x}{y_{OT}(0)+t\dot{y}_{OT}(0)}\text{, for any } t \text{ in } [0, T_f]\,.$$

$$\Leftrightarrow \left(x+t\dot{x}-\frac{t^2}{2}\gamma_x\right)\left[y_{OT}(0)+t\dot{y}_{OT}(0)\right] = \left[x_{OT}(0)+t\dot{x}_{OT}(0)-\frac{t^2}{2}\gamma_x\right]\left(y+t\dot{y}\right) \quad , \forall\, t \in [0, T_f]\,.$$

After reordering the terms of this equation, we get

$$\begin{cases} x_{OT}(0)y = y_{OT}(0)x \\ x_{OT}(0)\dot{y}+\dot{x}_{OT}(0)y = \dot{y}_{OT}(0)x+y_{OT}(0)\dot{x} \\ \dot{x}_{OT}(0)\dot{y}-\frac{\gamma_x}{2}y = \dot{y}_{OT}(0)\dot{x}-\frac{\gamma_x}{2}y_{OT}(0) \\ \dot{y} = \dot{y}_{OT}(0) \end{cases}$$

We end up with the system

$$\mathbf{M}X = B \qquad (\Sigma)$$

with $\mathbf{M}=\begin{bmatrix} y_{OT}(0) & -x_{OT}(0) & 0 & 0 \\ \dot{y}_{OT}(0) & -\dot{x}_{OT}(0) & y_{OT}(0) & 0 \\ 0 & -\frac{\gamma_x}{2} & -\dot{y}_{OT}(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and

$$B=\begin{bmatrix} 0 \\ x_{OT}(0)\dot{y}_{OT}(0) \\ -\dot{x}_{OT}(0)\dot{y}_{OT}(0)-\frac{\gamma_x}{2}y_{OT}(0) \\ \dot{y}_{OT}(0) \end{bmatrix}$$

We note that X_{OT} is a solution of (Σ).

The vector X_{OT} is the unique solution of (Σ) if and only if $\det(\mathbf{M})\neq 0$, in other words, the trajectory of the target is observable. Conversely, the trajectory of the target is not observable if and only if $\det(\mathbf{M})=0$. Consequently, the discussion is about $\det(\mathbf{M})$.

We readily get

$$\det(\mathbf{M}) = -x_{OT}(0)\dot{y}_{OT}^2(0)+y_{OT}(0)x_{OT}(0)\dot{y}_{OT}(0)+\frac{1}{2}y_{OT}^2(0)\gamma_x\,.$$

If $\det(\mathbf{M})=0$, we have to discriminate the subcase where $y_{OT}(0)\neq 0$ from the subcase $y_{OT}(0)=0$.

If $y_{OT}(0)=0$, then $x_{OT}(0)=0$. The case $x_{OT}(0)=0$ means that the target and the observer are co-localized at the initial time. This does not satisfy the assumptions given in II A. This case is hence discarded.

If $y_{OT}(0)\neq 0$, then $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0)-\dot{x}_{OT}(0)y_{OT}(0)\right]$; that is, the observer and the target are on an RDVR-II (cf. Proposition 3). Since the acceleration γ_x is not equal to zero, $\dot{y}_{OT}(0)$ cannot be equal to zero. As the consequence, the first two columns of $\mathbf{M}$ are not collinear. We remark also that the fourth column of the matrix $\mathbf{M}$ cannot be simultaneously collinear with another one, so the rank of $\mathbf{M}$ is equal to 3.

The set of solutions of (Σ) is the set of the vectors defined by $X_{OG}=X_{OT}+\alpha\Xi$, where α is a scalar and $\Xi=[\xi_1 \quad \xi_2 \quad \xi_3 \quad 0]^{\mathrm{T}}$ is a nonzero vector in the null space of $\mathbf{M}$. We verify that $\theta(t)$ has a special form: $\theta(t)=\tan^{-1}(\mu_0+\mu_1 t)$, with $\mu_0=\frac{x_{OT}(0)}{y_{OT}(0)}$ and $\mu_1=\frac{y_{OT}(0)\dot{x}_{OT}(0)-\dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$.

Note that $\frac{x_{OG}(0)+t\dot{x}_{OG}(0)-\frac{t^2}{2}\gamma_x}{y_{OG}(0)+t\dot{y}_{OG}(0)}=\mu_0+\mu_1 t$ is equivalent to

$$x_{OG}(0)+t\dot{x}_{OG}(0)-\frac{t^2}{2}\gamma_x=(\mu_0+\mu_1 t)(y_{OG}(0)+t\dot{y}_{OG}(0))\,.$$

Hence, at time $t_c=-\frac{y_{OG}(0)}{\dot{y}_{OG}(0)}=-\frac{y_{OG}(0)}{\dot{y}_{OT}(0)}$ (which depends on the relative coordinate $y_{OG}(0)$ of the considered ghost), we get $x_{OG}(t_c)=0$ and $y_{OG}(t_c)=0$. As a consequence, all the ghosts and the observer are on an RDVR-II (but at different times of rendezvous).

Case (2): $y_{OT}(0)=0$ and $\dot{y}_{OT}(0)=0$; that is, $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear. Note that
O and T are not necessarily on an RDVR-I.

Then $\theta(t, X)=\theta(t, X_{OT}) = \pm\frac{\pi}{2} \quad \Leftrightarrow \quad y+t\dot{y}=0$; that is, $y=\dot{y}=0$.

The bearing rate is zero and the set of solutions is the line of sight of the target: any ghost traveling in this line (the X-axis) is detected in the same (constant) bearing $\left(\pm\frac{\pi}{2}\right)$ as the target of interest. The target's trajectory is not observable.

QED.

D. Observer in CV and then CA motion and the converse

Observability must be studied only when the target is not observable during the CA motion of the observer, that is, when O and T are on an RDVR of type II or the bearings are constant. To do this, we need the notion of an "angle-equivalent non-maneuvering observer" [7] and [12] when O is in CA motion; the existence of a virtual observer (E) in CV motion continuously collecting the same bearings as the observer will be proved in the sequel.

In the rest of the paragraph, we will assume that $\tan\theta(t)=\mu_0+\mu_1 t$ (during the CA motion only).

Lemma 1

If the observer and the target are on a rendezvous route of type II or the bearings are constant, then a virtual observer, located at $P_O(0)$ *at time* $t=0$*, in CV motion, continuously collecting the same bearings as the observer, exists.*

Proof:

We denote in this proof the initial position and the velocity of a virtual observer in CV motion by $P_E(0)=[x_E(0) \quad y_E(0)]^T=[x_O(0) \quad y_O(0)]^T$ and $V_E=[\dot{x}_E \quad \dot{y}_E]^T$, respectively.

If the bearings are constant, then $y_O(0)=\dot{y}_O(0)=0$. The virtual observer (in CV motion) such that $P_E(0)=P_O(0)$ and $V_E=V_O(0)$ collects the same bearings as O.

If O and T are on a rendezvous route of type II, suppose that a virtual observer E in CV motion exists. The equality $\theta(t)=\tan^{-1}\left(\frac{x_{OT}(0)+t\,\dot{x}_{ET}}{y_{OT}(0)+t\,\dot{y}_{ET}}\right)$ implies that $\frac{x_{OT}(0)+t\,\dot{x}_{ET}}{y_{OT}(0)+t\,\dot{y}_{ET}}=\mu_0+\mu_1 t$. Consequently $\dot{y}_{ET}=0$, and $\frac{\dot{x}_{ET}}{y_{OT}(0)}=\mu_1$. Since $\mu_1=\frac{y_{OT}(0)\dot{x}_{OT}(0)-\dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$ (see Proposition 10), we get $\dot{x}_{ET}=\dot{x}_{OT}(0)-\dot{y}_{OT}(0)\frac{x_{OT}(0)}{y_{OT}(0)}$. We end up with $V_E=\begin{bmatrix}\dot{x}_O(0)+\dot{y}_{OT}(0)\frac{x_{OT}(0)}{y_{OT}(0)}\\ \\ \dot{y}_T\end{bmatrix}$. Note that this virtual observer is unique.

QED.

Hereafter, we propose an example of a scenario where the observer has a higher order dynamic than the target and the target's trajectory still remains unobservable. We have chosen a scenario with an RDVR-II (hence satisfying Proposition 3): $P_O(0)=[0 \quad 0]^T$, $V_O(0)=[10 \quad 2]^T$ (m/s), $P_T(0)=[3000 \quad 4000]^T$ (m), and $V_T=[-6 \quad -7]^T$ (m/s) with $\gamma_x=-0.0416$ m/s^2. The duration is 6 minutes. Figure 6 depicts the maneuvering observer together with the target (thick lines) and four ghosts (thin lines). Moreover, the trajectory of the virtual observer E is plotted. Four lines of sight are given.

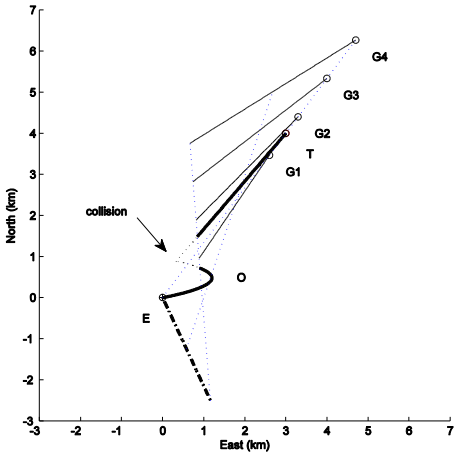

Figure 6. Non-observable trajectory in BOTMA, the target, some ghosts, and the bearing-equivalent-non-maneuvering observer.

Proposition 11: Observability criterion in BOTMA for CA-CV and CV-CA motions

Assume that the observer is successively in CV motion and in CA motion (or in CA motion and in CV motion). The target's trajectory is observable in BOTMA if and only if the bearing rate is non-null (that is, the bearings are not constant).

Proof:

We have only to consider the case where the target's trajectory is not observable during the maneuvering phase.

The first leg starts at time $t=0$ and ends at time $t=T_1$. At this time, the CA motion starts and finishes at time $t=T_f$.

If the bearings are constant during this phase, then $P_E(0)=P_O(T_1)$ and $V_E=V_O(0)$ (see lemma 1). We deduce that the bearings during $[0, T_1]$ are also constant, that is, the respective trajectories of the virtual observer and of the target are on the same line. The target is non observable.

If O and T are on an RDVR-II, following Lemma 1, the observer collects the same bearings as the non-maneuvering observer E. Hence, the observer collects the bearings acquired by an observer whose trajectory is composed of two legs: the first one is defined by $V_O(0)$, and the second one by V_E. Note that V_E cannot be equal to $V_O(0)$, otherwise $\dot{y}_T=\dot{y}_O(0)$ and consequently $\gamma_x=0$ (see Proposition 3). In short, the observer acquires the same bearings as the ones collected by a leg-by-leg maneuvering observer. Following [10], if the bearings are not constant, the target's trajectory is observable from the equivalent observer E. Proposition 6 completes the proof.

Proposition 7 allows us to determine when the observer is first in CA motion and then in CV motion.

QED.

IV. Observability Criteria in ROTMA

We proved in [13] that the target's trajectory is not observable in ROTMA when the observer is in CV motion. Actually, in

this case, the ROTMA problem has a linear version, similarly to BOTMA. Indeed, even if the range does not depend linearly on the vector X, the square of the range can be expressed in linear form, relative to another vector denoted Z:

$$r(t)=\|P_{OT}(0)+tV_{OT}\| \Leftrightarrow r^2(t)=\|P_{OT}(0)\|^2+2tV_{OT}^TP_{OT}(0)+t^2\|V_{OT}\|^2 \quad \forall t$$

We define $Z=\begin{bmatrix}\|P_{OT}(0)\|^2\\ V_{OT}^TP_{OT}(0)\\ \|V_{OT}\|^2\end{bmatrix}=\begin{bmatrix}Z_1\\Z_2\\Z_3\end{bmatrix}$.

So, $r^2(t_k)=\begin{bmatrix}1 & t_k & t_k^2\end{bmatrix}Z$, $k=1, 2, 3, \cdots$.

This is another proof of the non-observability of the target's trajectory in ROTMA, since this trajectory depends (mathematically speaking) on a three-dimensional vector, whereas the trajectory is defined by X. The vector Z is nevertheless observable: it defined the set of solutions given in Section IV-B of [13], which is an uncountable set (due to this linearity).

In the two next subsections, we will consider two types of kinematics of the observer: CT motion and CA motion.

A. Observer in CT motion

Proposition 12: Observability in ROTMA for CT motion
If the observer is traveling along an arc of a circle, then any target moving with a constant velocity is observable by range measurements only.

Proof:
Again, the proof is made in continuous time as in BOTMA. Suppose now that another target G moving with a constant velocity, say V_G, is at the same range as the target of interest T. The square of the range at any time t is

$$r^2(t)=[x_T(t)-x_O(t)]^2+[y_T(t)-y_O(t)]^2=[x_G(t)-x_O(t)]^2+[y_G(t)-y_O(t)]^2$$

$$\Leftrightarrow$$

$$[x_T(0)+t\dot{x}_T-\rho\sin(\omega t+\varphi)]^2+[y_T(0)+t\dot{y}_T-\rho\cos(\omega t+\varphi)]^2$$
$$=[x_G(0)+t\dot{x}_G-\rho\sin(\omega t+\varphi)]^2+[y_G(0)+t\dot{y}_G-\rho\cos(\omega t+\varphi)]^2$$
$$\forall t.$$

or, equivalently,

$$x_T^2(0)+y_T^2(0)+\rho^2+t^2(\dot{x}_T^2+\dot{y}_T^2)+2t[\dot{x}_Tx_T(0)+\dot{y}_Ty_T(0)]$$
$$-2\rho[x_T(0)\sin(\omega t+\varphi)+y_T(0)\cos(\omega t+\varphi)]$$
$$-2t\rho[\dot{x}_T\sin(\omega t+\varphi)+\dot{y}_T\cos(\omega t+\varphi)]$$
$$=\ x_G^2(0)+y_G^2(0)+\rho^2+t^2(\dot{x}_G^2+\dot{y}_G^2)+2t[\dot{x}_Gx_G(0)+\dot{y}_Gy_G(0)]$$
$$-2\rho[x_G(0)\sin(\omega t+\varphi)+y_G(0)\cos(\omega t+\varphi)]$$
$$-2t\rho[\dot{x}_G\sin(\omega t+\varphi)+\dot{y}_G\cos(\omega t+\varphi)],\forall t.$$

This implies the following five equalities:

$$\forall t\ \begin{cases}x_G^2(0)+y_G^2(0)=x_T^2(0)+y_T^2(0)\\ \dot{x}_G^2+\dot{y}_G^2=\dot{x}_T^2+\dot{y}_T^2\\ \dot{x}_Gx_G(0)+\dot{y}_Gy_G(0)=\dot{x}_Tx_T(0)+\dot{y}_Ty_T(0)\\ x_G(0)\sin(\omega t+\varphi)+y_G(0)\cos(\omega t+\varphi)\\ \qquad =x_T(0)\sin(\omega t+\varphi)+y_T(0)\cos(\omega t+\varphi) \quad (17)\\ \dot{x}_G\sin(\omega t+\varphi)+\dot{y}_G\cos(\omega t+\varphi)\\ \qquad =\dot{x}_T\sin(\omega t+\varphi)+\dot{y}_T\cos(\omega t+\varphi). \quad (18)\end{cases}$$

Equations (17) and (18) are equivalent to

$$\begin{bmatrix}\sin(\omega t+\varphi)\\ \cos(\omega t+\varphi)\end{bmatrix}^T[P_G(0)-P_T(0)]=0 \quad \forall t \text{ and}$$

$$\begin{bmatrix}\sin(\omega t+\varphi)\\ \cos(\omega t+\varphi)\end{bmatrix}^T[V_G-V_T]=0 \quad \forall t. \text{ Since } \begin{bmatrix}\sin(\omega t+\varphi)\\ \cos(\omega t+\varphi)\end{bmatrix} \text{ spans the}$$

whole two-dimensional space, $P_G(0)=P_T(0)$ and $V_G=V_T$.

QED.

Obviously, if the observer's trajectory contains at least one arc of a circle, then the trajectory of any target having a constant velocity is observable in ROTMA.

B. Observer in CA motion

We proved in [13] that, when the observer has a leg-by-leg motion, the trajectories of the ghosts are obtained by orthogonal transformations of the trajectory of the target of interest. The matrices of these transformations are shown to be constant (they are independent of time). They are denoted $\mathbf{H}$.

Here, the matrices of these transformations can depend on time. This makes the analysis much more complex. In the following subsection, two examples are given: one with a constant $\mathbf{H}$ and another with a non-constant matrix denoted $\mathbf{H}(t)$.

1) Examples of constant and non-constant orthogonal transformation matrices

First of all, we consider the case where $P_{OT}(t)$ is not collinear with Γ. Let us prove that a ghost-target exists. Indeed, consider the time-dependent vector defined by $P_{OG}(t)=\mathbf{H}\left[P_{OT}(0)+tV_{OT}(0)-\frac{t^2}{2}\Gamma\right]$ for some orthogonal (and constant) matrix $\mathbf{H}$. The equality $P_{OG}(t)=P_{OG}(0)+tV_{OG}(0)-\frac{t^2}{2}\Gamma$ holds if and only if $P_{OG}(0)=\mathbf{H}P_{OT}(0)$, $V_{OG}(0)=\mathbf{H}V_{OT}(0)$, and $\Gamma=\mathbf{H}\Gamma$. This last equality implies that $\mathbf{H}$ is the orthogonal matrix of the symmetry around the line spanned by the vector Γ. This constant orthogonal matrix will be denoted $\mathbf{S}$ in the sequel. Hence the vector $P_{OG}(t)$ defines the trajectory of a ghost-target.
We can conclude by the following proposition:

Proposition 13
If $P_{OT}(0)$ or $V_{OT}(0)$ is not collinear with Γ, then the target is not observable.

Let us present an example of this situation, depicted in Fig. 7.

At the beginning of the scenario, the observer starts at $[0 \quad 0]^T$ with an initial speed of 5 m/s and an initial heading of 90°. Its acceleration is the constant vector $\Gamma = [-0.006 \quad 0.002]^T$ (m/s^2). The target starts at $[0 \quad 15]^T$ (km) with a speed of 4 m/s and a heading of 0°. The duration of the simulation is 45 min. Note that with an ad-hoc rotation, we can be again in the case where $\gamma_y = 0$.

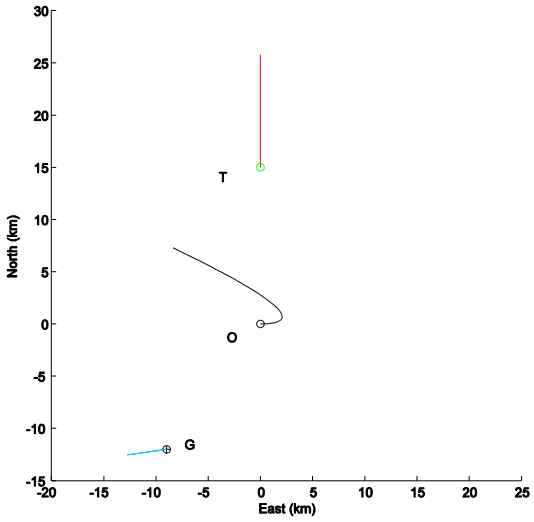

Figure 7. ROTMA: The observer in constant acceleration motion, the target, and a ghost.

Conversely, if $P_{OT}(0), V_{OT}(0)$, and Γ are collinear, then the target is not necessarily observable. Let us give a non-obvious example:

Let us consider an observer starting from $\begin{bmatrix}0\\0\end{bmatrix}$ with a certain velocity and the acceleration vector $\Gamma = \begin{bmatrix}-2\\0\end{bmatrix}$ and a target whose relative motion is defined by $P_{OT}(0) = \begin{bmatrix}u^2\\0\end{bmatrix}$ (for a certain value $u \neq 0$) and $V_{OT}(0) = \begin{bmatrix}0\\0\end{bmatrix}$. Then a ghost exists and its relative trajectory is defined by $P_{OG}(0) = \begin{bmatrix}-u^2\\0\end{bmatrix}$ and $V_{OG}(0) = \begin{bmatrix}0\\2u\end{bmatrix}$. We have $P_{OG}(t) = P_{OG}(0) + tV_{OG}(0) - \frac{t^2}{2}\Gamma$. We readily check that $P_{OG}(t) = \mathbf{H}(t)P_{OT}(t)$ for the rotation matrix

$$\mathbf{H}(t) = \frac{1}{u^2+t^2}\begin{bmatrix}-u^2+t^2 & -2ut\\ 2ut & -u^2+t^2\end{bmatrix}.$$

The second example leads us to think that analyzing observability must not be reduced to seeking constant orthogonal matrices, conversely to the case of a leg-by-leg trajectory of the observer [13].

2) Necessary and sufficient observability condition

The following analysis will be conducted for the relative motion with respect to the observer's trajectory. The question is to identify, if they exist, vectors $X = [x \quad y \quad \dot{x} \quad \dot{y}]^T$ such that

$$\left\| \begin{bmatrix}x + t\dot{x}\\ y + t\dot{y}\end{bmatrix} - \frac{t^2}{2}\Gamma \right\| = \|P_{OT}(t)\|, \quad \forall\, t \qquad (19)$$

Such a vector will be called a solution. The unknowns are the initial relative ghost-target position $P_{OG}(0) = [x \quad y]^T$ and its initial relative velocity $V_{OG}(0) = [\dot{x} \quad \dot{y}]^T$.

From the above subsection, we know that the set of solutions contains, of course, the vector defining the trajectory of the target, that is, $X_{OT} = [x_{OT}(0) \quad y_{OT}(0) \quad \dot{x}_{OT}(0) \quad \dot{y}_{OT}(0)]^T$ and the vector $(\mathbf{I}_2 \otimes \mathbf{S})X_{OT} = [x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^T$, where the symbol $\otimes$ denotes the Kronecker product and $\mathbf{I}_2$ is the 2×2 identity matrix. For convenience, the vector $(\mathbf{I}_2 \otimes \mathbf{S})X_{OT}$ is denoted $X_{OG}^{\mathbf{S}}$. It defines the trajectory of a ghost.

We will show below that most of the time, other ghost-targets exist and the vectors defining their trajectories are linked to X_{OT} by a time-dependent orthogonal matrix $\mathbf{I}_2 \otimes \mathbf{H}(t)$.

Before starting our analysis, we give a fundamental result in the following lemma: it establishes the equivalence between observability in discrete time and observability in continuous time. Therefore, we will conduct our analysis in continuous time, but the results will be valid for discrete time.

Lemma 2
If four different times τ_n in $[0, T_f]$, $n = 1, 2, 3, 4$, satisfy

$$\left\| P_{OG}(0) + \tau_n V_{OG}(0) - \frac{\tau_n^2}{2}\Gamma \right\| = \|P_{OT}(\tau_n)\| \qquad (20)$$

then for any t in $[0, T_f]$

$$\left\| P_{OG}(0) + tV_{OG}(0) - \frac{t^2}{2}\Gamma \right\| = \|P_{OT}(t)\|, \forall\, t \in [0, T_f]. \qquad (21)$$

Proof:
First of all, note that eq. (21) is equivalent to

$$\left\| P_{OG}(0) + tV_{OG}(0) - \frac{t^2}{2}\Gamma \right\|^2 = \|P_{OT}(t)\|^2 \qquad (22)$$

Each term of this equality is a polynomial function of degree 4:

$$\left\| P_{OG}(0)+tV_{OG}(0)-\frac{t^2}{2}\Gamma \right\|^2 = a_G + 2b_G\,t + c_G\,t^2 - d_G\,t^3 + \frac{\|\Gamma\|^2}{4}t^4$$

, with

$$a_G = \|P_{OG}(0)\|^2 = x^2 + y^2 \qquad (23-a)$$

$$b_G = P_{OG}(0)^{\mathrm{T}} V_{OG}(0) = x\dot{x} + y\dot{y} \qquad (23-b)$$

$$c_G = \|V_{OG}(0)\|^2 - P_{OG}(0)^{\mathrm{T}}\Gamma = \dot{x}^2 + \dot{y}^2 - x\gamma_x \qquad (23-c)$$

$$d_G = V_{OG}(0)^{\mathrm{T}}\Gamma = \dot{x}\gamma_x \qquad (23-d)$$

$\|P_{OT}(t)\|^2 = a_T + 2b_T\,t + c_T\,t^2 - d_T\,t^3 + \frac{\|\Gamma\|^2}{4}t^4$, with

$$a_T = \|P_{OT}(0)\|^2 = r_0^2 \qquad (24-a)$$

$$b_T = P_{OT}(0)^{\mathrm{T}} V_{OT}(0) = r_0\,v_r\cos(\theta_0 - h_r) \qquad (24-b)$$

$$c_T = \|V_{OT}(0)\|^2 - P_{OT}(0)^{\mathrm{T}}\Gamma = v_r^2 - r_0\,\gamma_x\sin\theta_0 \qquad (24-c)$$

$$d_T = V_{OT}(0)^{\mathrm{T}}\Gamma = v_r\,\gamma_x\sin h_r \qquad (24-d)$$

Eq. (20), which is equivalent to

$$\left\| P_{OG}(0)+\tau_n V_{OG}(0)-\frac{\tau_n^2}{2}\Gamma \right\|^2 = \|P_{OT}(\tau_n)\|^2 \;,\; n=1,\,2,\,3,\,4$$

gives us four instants where these two polynomial functions take the same values. Because they have a common coefficient (the 4th degree coefficient which is $\frac{\|\Gamma\|^2}{4}$), these two polynomial functions are equal.
QED.

Eq. (23) and (24) yield the following lemma on which the search of ghost-target is based:

Lemma 3
The set of solutions of (19) is defined by the following equations:

$$x^2 + y^2 = r_0^2 \qquad (25)$$

$$x\dot{x} + y\dot{y} = r_0\,v_r\cos(\theta_0 - h_r) \qquad (26)$$

$$\dot{x}^2 + \dot{y}^2 - x\gamma_x = v_r^2\cos^2 h_r - r_0\,\gamma_x\sin\theta_0 \qquad (27)$$

$$\dot{x}\gamma_x = v_r\gamma_x\sin h_r \qquad (28)$$

Note that eq (28) implies $\dot{x} = \dot{x}_{OT}(0)$.

The next proposition characterizes the set of ghost-targets. the set of solutions among the set of four-dimensional vectors whose first component is a root of a certain quadratic equation given below, and the third component is $\dot{x}_{OT}(0)$.

Proposition 14
The set of targets at the same range as the target of interest is composed by the target whose trajectory is defined by $X_{OG}^{\mathrm{S}} = (\mathbf{I_2}\otimes\mathbf{S})X_{OT}$ *and by those whose trajectories are defined by* $[x \quad y \quad \dot{x} \quad \dot{y}]^{\mathrm{T}}$, *such that*

a) x *is equal to* $x_{OT}(0)$ *or satisfies* $x^2 + \frac{v_r^2}{\gamma_x}x + r_0\frac{v_r^2}{\gamma_x}\sin(\theta_0 - 2h_r) - r_0^2 = 0$
b) and $\dot{x} = \dot{x}_{OT}(0)$.

Due to its length, the proof (based on Lemma 3) is given in Appendix A.

The two other components of X_{OG}, that is, y and $\dot{y}$, remain to be identified using Eqs. (25), (26), and (27).

For convenience, we now define $Q(x) = x^2 + \beta x + r_0\,\beta S - r_0^2$ with $\beta = \frac{v_r^2}{\gamma_x}$ and $S = \sin(\theta_0 - 2h_r)$. Note that β is negative. By convention, the heading h_r is zeroed when $v_r = 0$.

Now, we are able to give a necessary and sufficient observability condition.

Proposition 15: Observability condition in ROTMA for a CA motion
Assuming that the observer is traveling with a constant acceleration vector, the trajectory of the target is observable from at least four range measurements acquired at different times if and only if O *and* T *are on an RDVR-I.*

Proof:
Firstly, suppose that the trajectory of the target is observable. From Proposition 13, $P_{OT}(0)$, $V_{OT}(0)$, and Γ are collinear, so $P_{OT}(0) = \eta\Gamma$ and $V_{OT}(0) = \lambda\Gamma$. In terms of coordinates, this yields $x_{OT}(0) = \eta\gamma_x$, $\dot{x}_{OT}(0) = \lambda\gamma_x$, $y_{OT}(0) = 0$, and $\dot{y}_{OT}(0) = 0$. Hence

$\beta = \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$, $r_0 S = r_0\sin(\theta_0 \pm \pi) = -r_0\sin\theta_0 = -x_{OT}(0)$, and consequently

$$Q(x) = x^2 + \frac{\dot{x}_{OT}^2(0)}{\gamma_x}x - x_{OT}(0)\frac{\dot{x}_{OT}^2(0)}{\gamma_x} - x_{OT}^2(0).$$

Formally, the roots of $Q(x) = 0$ are $x = x_{OT}(0)$ and $x = -x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$. Since the trajectory of the target is observable, either $x_{OT}(0) = -x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$, in other words, $x_{OT}(0)$ is a double root (this corresponds to $\eta = -\frac{\lambda^2}{2}$), or $-x_{OT}(0) - \frac{\dot{x}_{OT}^2(0)}{\gamma_x}$ is an unacceptable physical solution; that is $\left[x_{OT}(0) + \frac{\dot{x}_{OT}^2(0)}{\gamma_x}\right]^2 > r_0^2$. Because $r_0^2 = x_{OT}^2(0)$, this inequality is equivalent to $\lambda^2 > -2\eta$. The target and the

observer are on an RDVR-I from Proposition 2.

Conversely, suppose that the target and the observer are on an RDVR-I; that is $P_{OT}(0)=\eta\Gamma$ and $V_{OT}(0)=\lambda\Gamma$, and $\lambda^2 \geq -2\eta$. We have $[x_{OT}(0) \quad y_{OT}(0) \quad \dot{x}_{OT}(0) \quad \dot{y}_{OT}(0)]^{\mathrm{T}} = [\eta\gamma_x \quad 0 \quad \lambda\gamma_x \quad 0]^{\mathrm{T}}$. Under this assumption, the scalars a_T, b_T and , c_T, defined in (24), take the following values $a_T=\eta^2\gamma_x^2$, $b_T=\lambda\eta\gamma_x^2$, $c_T=(\lambda^2-\eta)\gamma_x^2$. So, the set of equations (25), (26), and (27) becomes

$$x^2+y^2=\eta^2\gamma_x^2 \qquad (29)$$

$$\lambda\gamma_x x+y\dot{y}=\lambda\eta\gamma_x^2 \qquad (30)$$

$$\lambda^2\gamma_x^2+\dot{y}^2-x\gamma_x=(\lambda^2-\eta)\gamma_x^2 \qquad (31)$$

which is equivalent to

$$y^2=\eta^2\gamma_x^2-x^2 \qquad (32)$$

$$y\dot{y}=\lambda\gamma_x(\eta\gamma_x-x) \qquad (33)$$

$$\dot{y}^2=\gamma_x(x-\eta\gamma_x) \qquad (34)$$

Taking the square of (33), we obtain $y^2\dot{y}^2=\lambda^2\gamma_x^2(\eta\gamma_x-x)^2$.

Then using (32) and (34), we get $\gamma_x(\eta^2\gamma_x^2-x^2)(x-\eta\gamma_x)=\lambda^2\gamma_x^2(\eta\gamma_x-x)^2$, which can be factorized into $-(\eta\gamma_x-x)^2(\eta\gamma_x+x)=\lambda^2\gamma_x(\eta\gamma_x-x)^2$ (35).

If $x\neq\eta\gamma_x$, then (35) is equivalent to $-(\eta\gamma_x+x)=\lambda^2\gamma_x$, from which we derive $x=-(\eta+\lambda^2)\gamma_x$. Substituting into (32), we have $y^2=\left[\eta^2-(\eta+\lambda^2)^2\right]\gamma_x^2$; that is, $y^2=-\lambda^2(2\eta+\lambda^2)\gamma_x^2$. As a consequence, $\lambda^2(2\eta+\lambda^2)\leq 0$. Let us consider the two following cases. (i) $\lambda=0$. Then $\eta\geq 0$ and $y=0$; we deduce from (29) that $x=-\eta\gamma_x$. Reporting this value into (34), we get $\dot{y}^2=-\eta\gamma_x^2$. We deduce that $\eta\leq 0$. Hence $\eta=0$, which is incompatible with the fact that $r(t)>0$, for $t\in[0,\ T_f\ ]$. (ii) $\lambda\neq 0$. Then $2\eta+\lambda^2\leq 0$, but by assumption, $2\eta+\lambda^2\geq 0$. Hence $2\eta+\lambda^2=0$. We have $y=0$. Now, from (33), we get $x=\eta\gamma_x$, which is impossible. From this discussion, we conclude that the case $x\neq\eta\gamma_x$ must be rejected.

Hence, $x=\eta\gamma_x$ and consequently $y=0$ and $\dot{y}=0$. The trajectory of the target is observable.

QED.

Recall that the FIM in BOTMA and the FIM in ROTMA are of the same rank. As a consequence, when O and T are on an RDVR-I, the FIM is singular (since in BOTMA, the target trajectory is unobservable). This is a nontrivial example of a mismatch between the singularity of the FIM and the observability (in ROTMA). A similar example was given in [13], when the observer is zigzagging.

3) Unobservable case: construction of the set of ghosts

When the observability condition given in Proposition14 is not satisfied, identifying the set of ghosts is interesting. We propose to solve this problem by using the same approach as previously, by exploiting the roots $(x_1,\ x_2)$ of $Q(x)$. To do this, we need a first result.

We start with the following proposition:

Proposition 16: Existence condition of only one ghost in ROTMA for a CA motion

Assume that the observer is in CA motion. One unique ghost exists if and only if O and T are on an RDVR-II. Moreover, its trajectory is defined by X_{OG}^{s}.

Proof:

First, assume that O and T are on an RDVR-II; that is, $x_{OT}(0)+t_c\dot{x}_{OT}(0)-\frac{1}{2}t_c^2\gamma_x=0$ and $y_{OT}(0)+t_c\dot{y}_{OT}(0)=0$.

Any ghost is also on an RDVR-II with the observer. Consequently, we have

$x+t_c\dot{x}-\frac{1}{2}t_c^2\gamma_x=0$ and $y+t_c\dot{y}=0$.

We deduce that $x+t_c\dot{x}=x_{OT}(0)+t_c\dot{x}_{OT}(0)$. Since $\dot{x}=\dot{x}_{OT}(0)$, we conclude that $x=x_{OT}(0)$, and $y=-y_{OT}(0)$ and $\dot{y}=-\dot{y}_{OT}(0)$.

We have one unique ghost given by $X=[x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^{\mathrm{T}}=X_{OG}^{\mathrm{s}}$.

Conversely, assume now that one unique ghost exists. We recall that its trajectory is defined by $X=[x \quad y \quad \dot{x} \quad \dot{y}]^{\mathrm{T}}$ and that $\dot{x}=\dot{x}_{OT}(0)$.

First of all, let us prove that $x=x_{OT}(0)$. Indeed, if $x\neq x_{OT}(0)$, then $y=\dot{y}=0$ (otherwise, another ghost exists whose trajectory is defined by $X'=[x \quad -y \quad \dot{x} \quad -\dot{y}]^{\mathrm{T}}$, which is in contradiction with the unicity of the ghost). At this point, we necessarily have $X=[x \quad 0 \quad \dot{x}_{OT}(0) \quad 0]^{\mathrm{T}}$. Suppose that $y_{OT}(0)\neq 0$ or $\dot{y}_{OT}(0)\neq 0$. Then, the vector $[x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^{\mathrm{T}}$ defines the trajectory of a ghost. Since the ghost is unique, we have $X=[x \quad 0 \quad \dot{x}_{OT}(0) \quad 0]^{\mathrm{T}}=[x_{OT}(0) \quad -y_{OT}(0) \quad \dot{x}_{OT}(0) \quad -\dot{y}_{OT}(0)]^{\mathrm{T}}$, which is a contradiction. Hence $y_{OT}(0)=\dot{y}_{OT}(0)=0$. To summarize, we have obtained $X_{OT}=[x_{OT}(0) \quad 0 \quad \dot{x}_{OT}(0) \quad 0]^{\mathrm{T}}$, and $X=[x \quad 0 \quad \dot{x}_{OT}(0) \quad 0]^{\mathrm{T}}$. Consequently, $x=x_{OT}(0)$,

which is in contradiction with the assumption. Hence, $x = x_{OT}(0)$.

Therefore, $Q(x_{OT}(0)) = 0$. Let us develop the expression of $Q(x_{OT}(0))$:

$$Q(x_{OT}(0)) = x_{OT}^2(0) + \beta\, x_{OT}(0) + r_0\, \beta S - r_0^2,$$

with $\beta = \frac{v_r^2}{\gamma_x}$ and

$$S = \sin(\theta_0 - 2h_r) = \sin\theta_0\left(\cos^2 2h_r - \sin^2 2h_r\right) - 2\cos\theta_0 \sin h_r \cos h_r.$$

$$Q(x_{OT}(0)) = -y_{OT}^2(0) + r_0 \sin\theta_0 \frac{v_r^2}{\gamma_x} + r_0 \frac{v_r^2}{\gamma_x}\left[\sin\theta_0\left(\cos^2 2h_r - \sin^2 2h_r\right) - 2\cos\theta_0 \sin h_r \cos h_r\right]$$

$$= -y_{OT}^2(0) + r_0 \sin\theta_0 \frac{v_r^2}{\gamma_x}\left(1 + \cos^2 2h_r - \sin^2 2h_r\right) - 2r_0 \frac{v_r^2}{\gamma_x}\cos\theta_0 \sin h_r \cos h_r$$

$$= -y_{OT}^2(0) + 2r_0 \sin\theta_0 \cos^2 2h_r \frac{v_r^2}{\gamma_x} - 2r_0 \frac{v_r^2}{\gamma_x}\cos\theta_0 \sin h_r \cos h_r$$

$$= -y_{OT}^2(0) + 2\frac{x_{OT}(0)}{\gamma_x}\dot{y}_{OT}^2(0) - 2\frac{y_{OT}(0)}{\gamma_x}\dot{x}_{OT}(0)\dot{y}_{OT}(0)$$

$$Q(x_{OT}(0)) = 0 \Leftrightarrow \gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right].$$

According to Proposition 3, the target and the observer are on an RDVR-II.

QED.

Here, the singularity of the FIM and the unobservability of the target's trajectory are consistent (see Proposition 9).

The following lemma provides us information about the two solutions of the quadratic equation $Q(x) = 0$, introduced in Proposition 14. This piece of information will help us to identify all the ghost-targets. We recall that $Q(x) = x^2 + \beta x + r_0\, \beta S - r_0^2$, with $\beta = \frac{v_r^2}{\gamma_x}$ and $S = \sin(\theta_0 - 2h_r)$.

Lemma 4

The equation $Q(x) = 0$ has two real roots defined by $x_1 = \frac{-\beta - \sqrt{\Delta}}{2}$ and $x_2 = \frac{-\beta + \sqrt{\Delta}}{2}$, with $\Delta = (\beta - 2r_0 S)^2 + 4r_0^2(1 - S^2)$. Moreover, $-r_0 \le x_1 \le r_0 \le x_2$.

Proof:

We readily verify that the discriminant of $Q(x) = 0$ is $\Delta = (\beta - 2r_0 S)^2 + 4r_0^2(1 - S^2)$, which is a positive quantity. Hence the two roots are real and are given by $x_1 = \frac{-\beta - \sqrt{\Delta}}{2}$ and $x_2 = \frac{-\beta + \sqrt{\Delta}}{2}$. Note that $x_1 \le x_2$.

We note that $Q(r_0) = \beta r_0 (S + 1)$, and $Q(-r_0) = \beta r_0 (S - 1)$. Hence, $Q(-r_0) \ge 0$ and $Q(r_0) \le 0$. We deduce that $-r_0 \le x_1 \le r_0 \le x_2$ and, as a consequence, that x_2 is strictly positive.

QED.

Lemma 4 will conduct our study according to the following table:

Table I: Values of (x_1, x_2)

	x_1	x_2
Case 1	$x_1 = -r_0$	$x_2 = r_0$
Case 2	$-r_0 < x_1 < r_0$	$x_2 = r_0$
Case 3	$x_1 = r_0$	$x_2 = r_0$
Case 4	$x_1 = -r_0$	$x_2 > r_0$
Case 5	$-r_0 < x_1 < r_0$	$x_2 > r_0$
Case 6	$x_1 = r_0$	$x_2 > r_0$

Recall that $x_1 + x_2 = -\beta$ and $x_1 x_2 = r_0(\beta S - r_0)$. We deduce that Case 1 is the only case where $v_r = 0$.

Case 1: $x_1 = -r_0$ and $x_2 = r_0$

The sum of the roots is zero, that is $\beta = 0$ $(\Leftrightarrow v_r = 0)$; as a consequence, $X_{OT} = [\, r_0 \sin\theta_0 \quad r_0 \cos\theta_0 \quad 0 \quad 0\,]^{\mathrm{T}}$.

From (25), for $x = x_1$ or $x = x_2$, we get $y = 0$.

For $x = x_1$, from (27), then $\dot{y}^2 + r_0\, \gamma_x = -r_0\, \gamma_x \sin\theta_0$, or equivalently $\dot{y}^2 = -r_0\, \gamma_x (1 + \sin\theta_0)$. Hence $\dot{y} = \pm\sqrt{-r_0\, \gamma_x (1 + \sin\theta_0)}$. We get two ghosts, given by $X = \begin{bmatrix} -r_0 \\ 0 \\ 0 \\ \pm\sqrt{-r_0\, \gamma_x (1 + \sin\theta_0)} \end{bmatrix}$. If $\theta_0 = -\frac{\pi}{2}$, then these two solutions merge into one: $X = [-r_0 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}} = X_{OT}$.

For $x = x_2$, from (27), then $\dot{y}^2 = r_0\, \gamma_x (1 - \sin\theta_0)$.

But the right member of this equation is negative or null (when $\theta_0 = \frac{\pi}{2}$), whereas the left member is positive or null. We get one solution if $\theta_0 = \frac{\pi}{2}$: $X = \begin{bmatrix} r_0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} = X_{OT}$.

In conclusion, in Case 1, if $\theta_0 \neq \pm\frac{\pi}{2}$, we end up with three ghosts, whose respective trajectories are defined by $X = \begin{bmatrix} -r_0 \\ 0 \\ 0 \\ \pm\sqrt{-r_0\,\gamma_x(1+\sin\theta_0)} \end{bmatrix}$ and X_{OG}^{s}.

If $\theta_0 = \frac{\pi}{2}$, we get two ghost-trajectories given by $X = \begin{bmatrix} -r_0 \\ 0 \\ 0 \\ \pm\sqrt{-2\,r_0\,\gamma_x} \end{bmatrix}$.

The target is observable if and only if $\theta_0 = -\frac{\pi}{2}$.

Case 2: $-r_0 < x_1 < r_0$ and $x_2 = r_0$

In this case, $x_1 r_0 = r_0(\beta S - r_0)$ and $x_1 + r_0 = -\beta$. Then necessarily, $S = -1$; that is, $\theta_0 = 2h_r - \frac{\pi}{2} \pmod{2\pi}$. Note that in this case, $\frac{1+\sin\theta_0}{2} = \sin^2 h_r$ and $\frac{1-\sin\theta_0}{2} = \cos^2 h_r$.

Since $x_2 - x_1 > 0$, we get $\beta + 2r_0 > 0$.

The root $x = x_1$ allows us to construct two ghosts: if $\sin h_r > 0$, then

$$X = \begin{bmatrix} -\beta - r_0 \\ \varepsilon\sqrt{-\beta(\beta+2r_0)} \\ \dot{x}_{OT}(0) \\ \varepsilon\sqrt{-\gamma_x(\beta+2r_0)\sin^2 h_r} \end{bmatrix}, \text{ with } \varepsilon = \pm 1; \text{ else}$$

$$X = \begin{bmatrix} -\beta - r_0 \\ \varepsilon\sqrt{-\beta(\beta+2r_0)} \\ \dot{x}_{OT}(0) \\ -\varepsilon\sqrt{-\gamma_x(\beta+2r_0)\sin^2 h_r} \end{bmatrix}.$$

For $x = x_2$, from (27) we get $\dot{x}^2 + \dot{y}^2 = v_r^2 - r_0\,\gamma_x \sin\theta_0 + r_0\,\gamma_x = v_r^2 + r_0\,\gamma_x(1-\sin\theta_0)$, which is equivalent to

$$\dot{y}^2 = v_r^2\cos^2 h_r + r_0\,\gamma_x(1-\sin\theta_0) = \frac{v_r^2}{2}(1+\cos 2h_r) + r_0\,\gamma_x(1-\sin\theta_0).$$

We have $\cos 2h_r = -\sin\theta_0$. So, we get

$$\begin{aligned} \dot{y}^2 &= \frac{v_r^2}{2}(1-\sin\theta_0) + r_0\,\gamma_x(1-\sin\theta_0) \\ &= \gamma_x\left[\frac{\beta}{2}(1-\sin\theta_0) + r_0(1-\sin\theta_0)\right] \\ &= \frac{1}{2}\gamma_x(\beta+2r_0)(1-\sin\theta_0) \\ &= \gamma_x(\beta+2r_0)\cos^2 h_r. \end{aligned}$$

The square $\dot{y}^2$ being positive or null and the right hand side term being negative or null, we have $\theta_0 = \frac{\pi}{2}$. That yields the vector $X = \begin{bmatrix} r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ 0 \end{bmatrix}$ (which is equal to the vector defining the trajectory of the target).

In conclusion, in case 2, if $\theta_0 \neq \frac{\pi}{2}$, then we have three ghosts; else we have two ghosts (in case 2, the target is not observable).

Case 3: $x_1 = x_2 = r_0$

In this case, $\Delta = 0$ or equivalently $S^2 = 1$ and $\beta = 2r_0 S$. Because β is negative, we get $S = -1$ (hence $\beta = -2S$).

First of all, note that $S = -1 \Leftrightarrow \theta_0 - 2h_r = -\frac{\pi}{2}$ and $\beta = -2r_0 \Leftrightarrow v_r = \sqrt{-2\,r_0\,\gamma_x}$.

From (25), we have $y = 0$. Reporting $x = r_0$ into (27) we get

$$\begin{aligned} &\dot{x}_{OT}^2(0) + \dot{y}^2 - r_0\,\gamma_x = v_r^2 - r_0\,\gamma_x\sin\theta_0 \\ \Leftrightarrow\quad &\dot{y}^2 = v_r^2 - \dot{x}_{OT}^2(0) - r_0\,\gamma_x\sin\theta_0 + r_0\,\gamma_x \\ &= \dot{y}_{OT}^2(0) + r_0(1-\sin\theta_0)\gamma_x \\ &= v_r^2\cos^2 h_r + r_0(1-\sin\theta_0)\gamma_x \\ &= \gamma_x\left[\beta\cos^2 h_r + r_0(1-\sin\theta_0)\right] \\ &= \gamma_x\left[\frac{\beta}{2}(1+\cos 2h_r) + r_0(1-\sin\theta_0)\right]. \\ &= \gamma_x\left[\frac{\beta}{2}(1-\sin\theta_0) + r_0(1-\sin\theta_0)\right] \\ &= \gamma_x\left[-r_0(1-\sin\theta_0) + r_0(1-\sin\theta_0)\right] \\ &= 0 \end{aligned}$$

Consequently, $\dot{y} = 0$.

We deduce a solution given by
$X = \begin{bmatrix} r_0 & 0 & \dot{x}_{OT}(0) & 0 \end{bmatrix}^{\mathrm{T}}$.

We verify that equations (25), (26), (27), and (28) are satisfied.

In conclusion, in Case 3, the target is observable if and only if $\theta_0 = \frac{\pi}{2}$ ($\Leftrightarrow h_r = \pm\frac{\pi}{2}$). Otherwise, two ghosts exist, defined by $X = \begin{bmatrix} r_0 & 0 & \dot{x}_{OT}(0) & 0 \end{bmatrix}^{\mathrm{T}}$ and X^{s}_{OG}.

Case 4: $x_1 = -r_0$ and $x_2 > r_0$.
As in case 2, we exploit the sum and the product of the roots: $-r_0\, x_2 = r_0(\beta S - r_0)$ and $x_2 - r_0 = -\beta$. Then necessarily, $S = 1$, which is equivalent to $\theta_0 - 2h_r = \frac{\pi}{2} \pmod{2\pi}$. In this case, $\frac{1+\sin\theta_0}{2} = \cos^2 h_r$.

We note that $\beta - 2r_0 < 0$. Again, from (25), we have $y = 0$.

From (27), we get $\dot{x}^2 + \dot{y}^2 = v_r^2 - r_0\,\gamma_x \sin\theta_0 - r_0\,\gamma_x = v_r^2 - r_0\,\gamma_x(1+\sin\theta_0)$, which is equivalent to $\dot{y}^2 = v_r^2\cos^2 h_r - r_0\,\gamma_x(1+\sin\theta_0)$
$= \frac{v_r^2}{2}(1+\cos 2h_r) - r_0\,\gamma_x(1+\sin\theta_0)$. We have $\cos 2h_r = \sin\theta_0$. So, we get

$$\begin{aligned}\dot{y}^2 &= \frac{v_r^2}{2}(1+\sin\theta_0) - r_0\,\gamma_x(1+\sin\theta_0)\\ &= \gamma_x\left[\frac{\beta}{2}(1+\sin\theta_0) - r_0(1+\sin\theta_0)\right],\\ &= \frac{1}{2}\gamma_x(\beta - 2r_0)(1+\sin\theta_0)\\ &= \gamma_x(\beta - 2r_0)\cos^2 h_r.\end{aligned}$$

which is a positive quantity.
So, two ghosts exist, defined by

$$X = \begin{bmatrix} -r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ \pm\sqrt{\gamma_x(\beta - 2r_0)\cos^2 h_r} \end{bmatrix}.$$

In conclusion, in Case 4, the target is observable if and only if $\theta_0 = -\frac{\pi}{2}$ ($\Leftrightarrow \cos h_r = 0$; that is $h_r = \pm\frac{\pi}{2}$). Otherwise, three ghosts exist:

$$X = \begin{bmatrix} -r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ \pm\sqrt{\gamma_x(\beta - 2r_0)\cos^2 h_r} \end{bmatrix} \text{ and } X^{\mathrm{s}}_{OG}.$$

Case 5: $-r_0 < x_1 < r_0$ and $x_2 > r_0$
From Eq. (25), $y = \pm\sqrt{r_0^2 - x_1^2}$. Introducing this value into (26), we get $\dot{y} = \pm\frac{r_0\, v_r\cos(\theta_0 - h_r) - x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2 - x_1^2}}$. This value also satisfies Eq. (27) (see Appendix A2). Consequently, we obtain the following two solutions:

$$X = \begin{bmatrix} x_1 \\ \varepsilon\sqrt{r_0^2 - x_1^2} \\ \dot{x}_{OT}(0) \\ \varepsilon\frac{r_0\, v_r\cos(\theta_0 - h_r) - x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2 - x_1^2}} \end{bmatrix}, \quad \text{with}$$

$\varepsilon = -1,\ +1$.
In conclusion, in Case 5, the target is never observable. At most, three ghosts exist, given by

$$X = \begin{bmatrix} x_1 \\ \varepsilon\sqrt{r_0^2 - x_1^2} \\ \dot{x}_{OT}(0) \\ \varepsilon\frac{r_0\, v_r\cos(\theta_0 - h_r) - x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2 - x_1^2}} \end{bmatrix}, \quad \text{with}$$

$\varepsilon = -1,\ +1$, and X^{s}_{OG}.
This case includes the scenarios where T and O are on an RDVR- II.

Case 6: $x_1 = r_0$ and $x_2 > r_0$
As in case 2, $r_0\, x_2 = r_0(\beta S - r_0)$ and $r_0 + x_2 = -\beta$. Necessarily, $S = -1$ and $x_2 = -\beta - r_0$. The equality $S = -1$ is equivalent to $\theta_0 = 2h_r - \frac{\pi}{2}$.

The assumption $x_2 > r_0$ implies that $\beta + 2r_0 < 0$; hence $v_r > \sqrt{-2r_0\gamma_x}$.

The component $\dot{y}$ is then readily computed:
$\dot{y} = \pm\sqrt{\frac{1}{2}\gamma_x(\beta + 2r_0)(1-\sin\theta_0)}$
$= \pm\sqrt{\gamma_x(\beta + 2r_0)\cos^2 h_r}$.

From (25), we get $y = 0$. So, two solutions exist,

defined by $X = \begin{bmatrix} r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ \pm\sqrt{\gamma_x(\beta + 2r_0)\cos^2 h_r} \end{bmatrix}$. The equations (25), (26), (27), and (28) are verified.

In conclusion, in Case 6, the target is observable if and only if $\theta_0 = \frac{\pi}{2} \Leftrightarrow h_r = \pm\frac{\pi}{2}$.

Else three ghosts exist and their trajectories are defined by $X = \begin{bmatrix} r_0 \\ 0 \\ \dot{x}_{OT}(0) \\ \pm\sqrt{\gamma_x(\beta + 2r_0)\cos^2 h_r} \end{bmatrix}$ *and* X_{OG}^{s}.

Table II: Summarized results of observability in ROTMA

	Necessary condition to have ghosts			
	0 (Observability case: RDVR-I)	*1*	*2*	*3*
Case 1	$\theta_0 = -\frac{\pi}{2}$	*never*	$\theta_0 = \frac{\pi}{2}$	$\theta_0 \neq \pm\frac{\pi}{2}$
Case 2	*never*	*never*	$\theta_0 = \frac{\pi}{2}$	$\theta_0 \neq \frac{\pi}{2}$
Case 3	$\theta_0 = \frac{\pi}{2}$	*never*	$\theta_0 \neq \frac{\pi}{2}$	*never*
Case 4	$\theta_0 = -\frac{\pi}{2}$	*never*	*never*	$\theta_0 \neq -\frac{\pi}{2}$
Case 5	*never*	*RDVR-II*	*never*	*Not RDVR-II*
Case 6	$\theta_0 = \frac{\pi}{2}$	*never*	*never*	$\theta_0 \neq \frac{\pi}{2}$

The content of Table II is consistent with Propositions 14, 15 and 16.

At this point, we can propose the following algorithm to construct (if existing), all the ghosts from a set of noise-free measurements $\{r_1, r_2, \cdots, r_K\}$, *with* r_k *the range at time* t_k*:*

If $r_k - \frac{1}{2}t_k^2\gamma_x = r_0 + t_k\dot{r}$, *or* $r_k + \frac{1}{2}t_k^2\gamma_x = r_0 + t_k\dot{r}$ *and* $\dot{r}^2 \geq -2r_0\gamma_x$, *then the trajectory of the target is observable and we compute the state vector by minimizing the criterion* $\sum_{k=1}^{K}\left[r(t_k, X) - r_k\right]^2$ *w.r.t.* X.

If not, we compute one of the vectors X *(it is not unique since the trajectory of the target is not observable) that minimizes* $\sum_{k=1}^{K}\left[r(t_k, X) - r_k\right]^2$, *then we choose this vector as the state vector of the target of interest (of course in reality, this choice can be wrong); this choice provides us with the corresponding polynomial function* $Q(x)$ *and its two roots. Finally, we* compute the other solutions X by exploiting the above analysis (we only have to identify in which case we are). Of course, we are unable at the end to *separate the wheat from the chaff*, that is, to identify the state vector of the true target among the ghost-targets.

4) Examples

In this paragraph, we give one example for each case. In the following scenarios, the observer has the same trajectory: at the very beginning, it starts from $\begin{bmatrix}0 & 0\end{bmatrix}^T$ with an initial velocity of $\begin{bmatrix}10 & 2\end{bmatrix}^T$ m/s. Its acceleration is the constant vector $\Gamma = \begin{bmatrix}-0.0416 & 0\end{bmatrix}^T$ (m/s²). The duration of the scenario is equal to 6 min ($T_f = 360$ s). The observer's trajectory is hence the same as in Section III. C. (as shown in Fig. 6).

In the next figures, the trajectories of the observer and target are shown by thick lines, whereas those of the ghosts are shown by thin lines. The capital letters ("O" for observer, "T" for target, and "G" for ghost) designate the moving objects.

a) Example of observable case (case 1)

The target starts at $\begin{bmatrix}-4000 & 0\end{bmatrix}^T$ (m) with a velocity of $\begin{bmatrix}10 & 2\end{bmatrix}^T$ (m/s). The observer and the target are on an RDVR-I, since $r(t) = r_0 - \frac{1}{2}\gamma_x t^2$ (Proposition 4 is satisfied). Fig. 8. depicts the scenario.

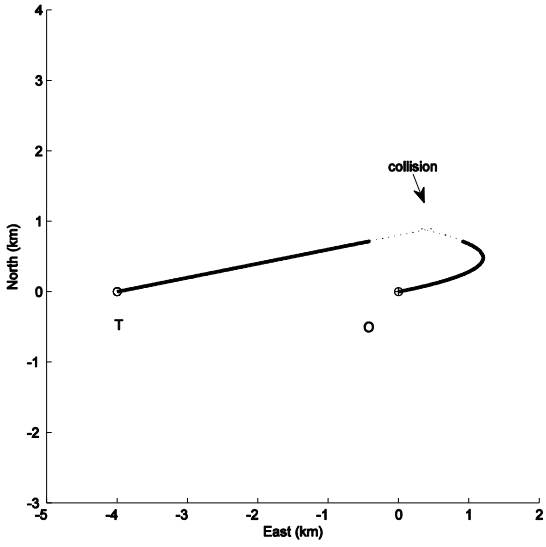

Figure 8. Observable case in ROTMA, RDVR-I.

b) Examples of unobservable cases

The following example corresponds to case 5 with a RDVR-II. The target starts at $\begin{bmatrix}3000 & 4000\end{bmatrix}^T$ (m) with a velocity of $\begin{bmatrix}-6 & -7\end{bmatrix}^T$ (m/s). We can readily verify that

$\tan\theta(t) = -0.0023t + 0.75$ thanks to Proposition 5. The observer and the target are on a rendezvous route of type II. We can see in Fig. 9. that one ghost exists, and it is in a rendezvous route with the observer (see Proposition 16).

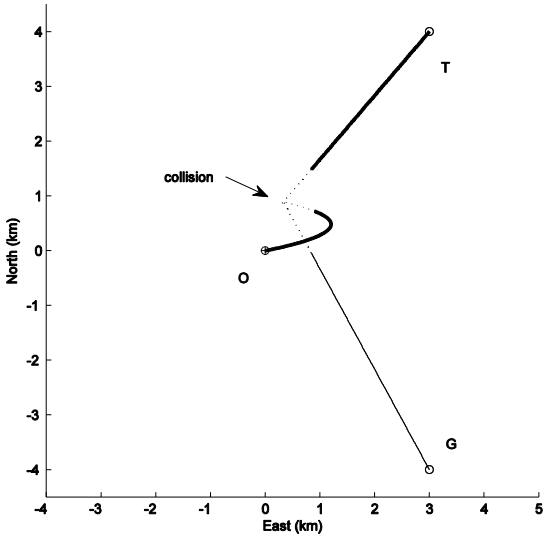

Figure 9. Unobservable case in ROTMA, RDVR-II.

The next example corresponds to case 2.

The target starts at $[4000 \quad 0]^T$ (m) with a velocity of $[25 \quad 2]^T$ (m/s). We can check that $r(t) + \frac{1}{2}\gamma_x t^2 = 4000 + 15t$, but $(225 =)\dot{r}^2 < -2r_0\gamma_x (= 333)$. Consequently, the observer and the target are not on a rendezvous route from Proposition 4. The bearings are constant (their value is 90°), two ghosts exist, as announced in Table II (See Fig. 10). Note that if the role of T and G1 (for example) are inverted (G1 becomes the target and T becomes a ghost), then this scenario corresponds to case 3: actually the polynomial function $Q(x)$ is not the same (see eq. (A6) in the proof of Proposition 14). The bearings of this new target are not constant, but two ghosts only exist.

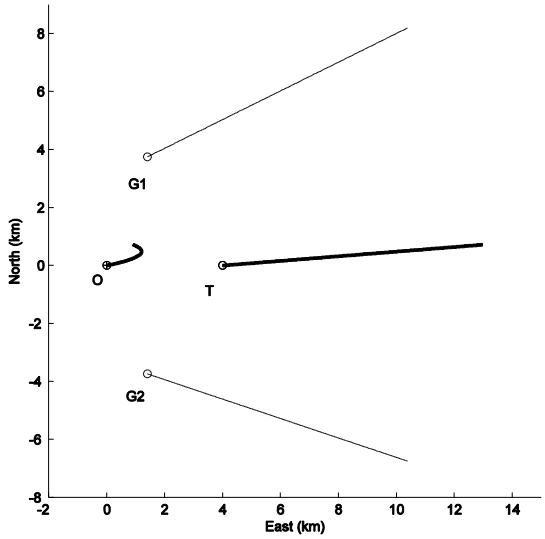

Figure 10. Unobservable case in ROTMA, with constant bearings and no rendezvous route.

The last example offers a scenario for which three ghosts exist. It corresponds to case 5.

The target starts at $[2000 \quad 3464]^T$ (m) with a velocity of $[14.6 \quad 16.3]^T$ (m/s). The observer and the target are not on a rendezvous route and the bearings are not constant. Three ghosts exist (the maximum number of ghosts), as depicted in Fig. 11.

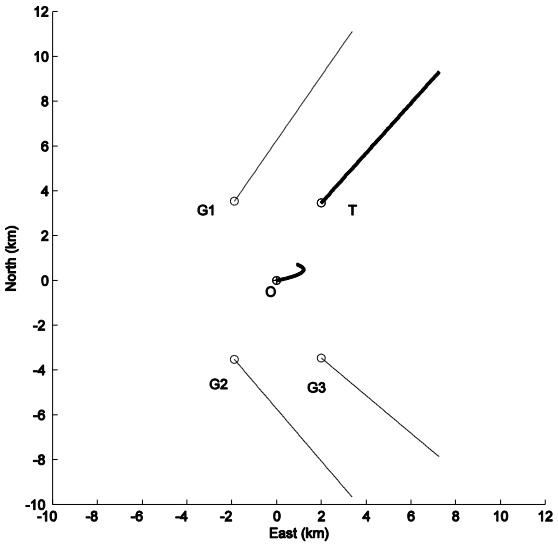

Figure 11. Unobservable case in ROTMA, general case (no rendezvous route and non-constant bearings).

V. CONCLUSION

In the present paper, the results of the observability analysis in ROTMA started in [13] for a target in constant velocity motion and an observer moving leg by leg, have been extended to a smooth observer's maneuver (constant turn motion and constant acceleration motion).

When the observer is in CT motion (see for example [1] and [4] in BOTMA and [14] and [15] in ROTMA), observability is guaranteed in BOTMA and in ROTMA as well. If a part of the displacement of the observer is in an arc of a circle, this conclusion remains valid.
When the observer is in CA motion, observability in BOTMA is guaranteed if the observer and the target are not in an RDVR and the bearings are not constant. In ROTMA, if they are in an RDVR and the bearings are constant, the trajectory of the target is observable. This proves that even if the observer kinematic is of an order greater than the kinematic of the target, observability is not guaranteed. This is not in contradiction with [5], whose authors established a necessary (but non-sufficient) observability condition in this case for BOTMA. When the target is not observable, in ROTMA, the set of ghost-targets is finite; we give the way to construct them from noise-free measurements and we end up with three ghost-targets at most. In BOTMA, the set of ghost-targets is uncountable. Measurement-based criteria that allow it to be known whether or not the target is observable are also given.
To summarize, unlike what we established in [13] (the observer trajectory was composed of several legs), that is, that the target is bearings-only observable if it is range-only unobservable and the converse, for the two types of smooth maneuvers considered here, this duality is not verified. All our results are summarized in the following Table:

Table III: Synthetic results about observability

Observer's kinematics	BOTMA	ROTMA
CT	Yes	Yes
CA: RDVD-I	No	Yes
CA: RDVD-II	No	No
CA: constant bearings	No	No
CA: others	Yes	No

We extended in BOTMA our analysis when the observer's trajectory is composed of a CA motion followed by a CV motion (and inversely): arguing fundamental properties, we proved that if the bearing rate is non-null, the target is observable.
Despite the rank equality of the FIMs in ROTMA and BOTMA (see [13]), the statuses of observability in ROTMA and in BOTMA are not necessarily the same. Even if this result is surprising, it is consistent with the theory (see for example [8]): the existence of a linear form of the BOTMA problem explains: (i) the equivalence between the regularity of the FIM and observability and (ii) the unaccountability of the set of ghost-targets in non-observable situations. Conversely, in ROTMA, in the case of non-observability, the FIM can be singular or not and the set of ghost-targets is finite. This is a proof that the problem of ROTMA cannot be expressed under a linear form (otherwise the set of ghost-targets would be a linear subspace and hence uncountable).
Obviously, we do not claim to have achieved a complete study of observability in ROTMA; for example, the cases when the observer's trajectory is composed of a CA motion followed by a CV motion, or when the motion of the observer is polynomial of order greater than two, or when the observer does not maneuver whereas the target does (which we studied in BOTMA in [2] and [9]), and so on must be investigated. Our study can help as a basis.

The next step (and paper) concerning ROTMA is the estimation, when the measurements are polluted by an additive noise. The observability analysis presented here will allow us to anticipate several difficulties: (i) thanks to the given criteria, we will be able to construct a statistical test to decide whether the observer and the target are on a rendezvous route or not. (ii) Moreover, for a given solution, we will construct the set of ghost-targets in the case of a lack of observability. (iii) Finally, an initialization of any numerical routine (necessary to compute the maximum likelihood estimate of the state vector), based on this analysis, can be proposed.

APPENDIX

A. *Proof of Proposition 14*

We exploit here the four equations of Lemma 3. In order to render the proof lighter, we drop the subscript T in a_T, b_T and c_T defined in (24): $a = r_0^2$, $b = r_0 v_r \cos(\theta_0 - h_r)$, and $c = v_r^2 - r_0 \gamma_x \sin\theta_0$.
From (28), we have $\dot{x} = \dot{x}_{OT}(0)$. Reporting this first result into (27), we get $\dot{x}_{OT}^2(0) + \dot{y}^2 - x\gamma_x = c$. Then, multiplying both sides of this equation by y^2, we get
$y^2\dot{x}_{OT}^2(0) + y^2\dot{y}^2 - x y^2\gamma_x = cy^2$. (A1)
From (26), we know that $y^2\dot{y}^2 = [b - x\dot{x}_{OT}(0)]^2$. Inserting this into (A1), we get
$y^2\dot{x}_{OT}^2(0) + (b - x\dot{x})^2 - x y^2\gamma_x = cy^2$. (A2)

Finally, using Eq. (25), we replace y^2 by $a - x^2$ in (A2), and we end up with the following equation: $(a - x^2)\dot{x}_{OT}^2(0) + [b - x\dot{x}_{OT}(0)]^2 - x(a - x^2)\gamma_x = c(a - x^2)$. (A3)

The cubic equation (A3) has at most three real roots (one of them is $x_{OT}(0)$). Let us denote the three solutions as x_i $(i = 0, 1, 2)$ (in some cases, only one or two roots exist). For convenience, the root $x_{OT}(0)$ will be denoted x_0.

To compute the two other roots, we first develop (A3):
$x^3\gamma_x + x^2[-\dot{x}_{OT}^2(0) + \dot{x}_{OT}^2(0) + c] + x[-2b\dot{x}_{OT}(0) - a\gamma_x] + \text{cst} = 0$

$\Leftrightarrow \quad x^3\gamma_x + x^2 c - x[2b\dot{x}_{OT}(0) + a\gamma_x] + \text{cst} = 0$. (A4)

Since x_0 is a root, we have
$x_0^3\gamma_x + x_0^2 c - x_0[2b\dot{x}_{OT}(0) + a\gamma_x] + \text{cst} = 0$. (A5)
The difference (A4) – (A5) is $(x^3 - x_0^3)\gamma_x + (x^2 - x_0^2)c - (x - x_0)[2b\dot{x}_{OT}(0) + a\gamma_x] = 0$, or, equivalently,

$(x-x_0)\{(x^2+x_0^2+x\,x_0)\gamma_x+(x+x_0)c-[\,2b\,\dot{x}_{OT}(0)+a\,\gamma_x\,]\}=0$. (A6)

Dividing (A6) by $(x-x_0)$, we get $(x^2+x_0^2+x\,x_0)\gamma_x+(x+x_0)c-[\,2b\,\dot{x}_{OT}(0)+a\,\gamma_x\,]=0$ (A7), which is a quadratic equation.

Rearranging the terms of (A7), we end up with
$x^2\gamma_x+x(x_0\gamma_x+c)+x_0^2\gamma_x+x_0\,c-[\,2b\,\dot{x}_{OT}(0)+a\,\gamma_x\,]=0$
or equivalently

$$x^2+x\frac{x_0\gamma_x+c}{\gamma_x}+x_0\frac{x_0\gamma_x+c}{\gamma_x}-\left[\,2\frac{b}{\gamma_x}\dot{x}_{OT}(0)+a\,\right]=0\cdot$$

Now, we replace the terms a, b, and c by their respective values. The equation to be solved is hence

$$x^2+x\frac{v_r^2}{\gamma_x}+r_0\frac{v_r^2}{\gamma_x}\sin\theta_0-2r_0\frac{v_r^2}{\gamma_x}\cos(\theta_0-h_r)\sin h_r-r_0^2=0$$

$$\Leftrightarrow\quad x^2+x\frac{v_r^2}{\gamma_x}+r_0\frac{v_r^2}{\gamma_x}[\sin\theta_0-2\cos(\theta_0-h_r)\sin h_r]-r_0^2=0$$

$$\Leftrightarrow\quad x^2+x\frac{v_r^2}{\gamma_x}+r_0\frac{v_r^2}{\gamma_x}\sin(\theta_0-2h_r)-r_0^2=0\cdot$$

QED.

B. *Case 5: Compatibility of Eqs. (26) and (27)*

We start this proof with two useful lemmas.

Lemma L1: $\cos^2(\theta_0-h_r)+\sin(\theta_0-2h_r)\sin\theta_0-\cos^2 h_r=0$.

Proof:

$$\cos^2(\theta_0-h_r)=\frac{1}{2}+\frac{1}{2}\cos(2\theta_0-2h_r)$$

$$\sin(\theta_0-2h_r)\sin\theta_0=-\frac{1}{2}[\cos(2\theta_0-2h_r)-\cos 2h_r]$$

$$-\cos^2 h_r=-\frac{1}{2}-\frac{1}{2}\cos 2h_r$$

Hence, $\cos^2(\theta_0-h_r)+\sin(\theta_0-2h_r)\sin\theta_0-\cos^2 h_r=0$

QED.

Lemma L2: If x_1 is a root of $Q(x)$, then
$r_0^2-x_1^2=\beta\,x_1+r_0\,\beta\,S$

Proof:

$Q(x)=x^2+\beta\,x+r_0\,\beta\,S-r_0^2$ with $\beta=\dfrac{v_r^2}{\gamma_x}$ and
$S=\sin(\theta_0-2h_r)=\sin\theta_0-2\cos(\theta_0-h_r)\sinh_r$

As x_1 is a root of $Q(x)$, we have

$$Q(x_1)=x_1^2+\beta\,x_1+r_0\,\beta\,S-r_0^2=0\cdot$$

Hence, $r_0^2-x_1^2=\beta\,x_1+r_0\,\beta\,S$.

QED.

Now, we are able to prove that
$\dot{y}=\pm\dfrac{r_0\,v_r\cos(\theta_0-h_r)-x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2-x_1^2}}$ satisfies (27); that is,
$\dot{y}^2=v_r^2\cos^2 h_r+\gamma_x(x_1-r_0\sin\theta_0)$.

To do this, we compute the difference

$$\left[\frac{r_0\,v_r\cos(\theta_0-h_r)-x_1\dot{x}_{OT}(0)}{\sqrt{r_0^2-x_1^2}}\right]^2-(v_r^2\cos^2 h_r+\gamma_x x_1-\gamma_x r_0\sin\theta_0)$$

or equivalently

$$[r_0\,v_r\cos(\theta_0-h_r)-x_1\dot{x}_{OT}(0)]^2-(r_0^2-x_1^2)(v_r^2\cos^2 h_r+\gamma_x x_1-\gamma_x r_0\sin\theta_0)=D\,.$$

First of all,

$$\begin{aligned}&[r_0\,v_r\cos(\theta_0-h_r)-x_1\dot{x}_{OT}(0)]^2\\ =\;& r_0^2\,v_r^2\cos^2(\theta_0-h_r)+x_1^2\dot{x}_{OT}^2(0)\\ &-2x_1\dot{x}_{OT}(0)\,r_0\,v_r\cos(\theta_0-h_r)\\ =\;& r_0^2\,v_r^2\cos^2(\theta_0-h_r)+x_1^2v_r^2\sin^2 h_r\\ &-2x_1\,r_0\,v_r^2\cos(\theta_0-h_r)\sin h_r\\ =\;& r_0^2\,v_r^2\cos^2(\theta_0-h_r)+x_1^2v_r^2\sin^2 h_r+x_1\,r_0\,v_r^2(S-\sin\theta_0)\end{aligned}\quad\text{(A8)}$$

Now we compute $(r_0^2-x_1^2)(v_r^2\cos^2 h_r+\gamma_x x_1-\gamma_x r_0\sin\theta_0)$. We use Lemma L2: $r_0^2-x_1^2=\beta\,x_1+r_0\,\beta\,S$.

Hence,

$$\begin{aligned}&(r_0^2-x_1^2)(v_r^2\cos^2 h_r+\gamma_x x_1-\gamma_x r_0\sin\theta_0)\\ &\quad=\beta(x_1+r_0\,S)(v_r^2\cos^2 h_r+\gamma_x x_1-r_0\gamma_x\sin\theta_0)\\ &\quad=(x_1+r_0\,S)(\beta\,v_r^2\cos^2 h_r+v_r^2x_1-v_r^2r_0\sin\theta_0)\end{aligned}$$

$$\begin{aligned}&=v_r^2x_1^2+x_1\,r_0\,v_r^2\,S-x_1\,r_0\,v_r^2\sin\theta_0\\ &\quad+(x_1+r_0\,S)\beta\,v_r^2\cos^2 h_r-r_0^2\,v_r^2S\sin\theta_0\\ &=v_r^2x_1^2+x_1r_0\,v_r^2(S-\sin\theta_0)+v_r^2\cos^2 h_r(x_1\beta+r_0\,S\beta)-r_0^2v_r^2S\sin\theta_0\end{aligned}\quad\text{(A9)}$$

We end up with $D=$(A8) – (A9), that is,

$$\begin{aligned}D=\;& r_0^2\,v_r^2\cos^2(\theta_0-h_r)+x_1^2v_r^2\sin^2 h_r+x_1\,r_0\,v_r^2(S-\sin\theta_0)\\ &-[v_r^2x_1^2+r_0\,v_r^2\,x_1(S-\sin\theta_0)+v_r^2\cos^2 h_r(x_1\beta+r_0\,S\beta)-r_0^2S\,v_r^2\sin\theta_0]\end{aligned}$$

$$\begin{aligned}&=r_0^2\,v_r^2\cos^2(\theta_0-h_r)+x_1^2v_r^2(\sin^2 h_r-1)\\ &\quad-v_r^2\cos^2 h_r(x_1\beta+r_0\,S\beta)+r_0^2v_r^2S\sin\theta_0\end{aligned}$$

$$\begin{aligned}&=r_0^2\,v_r^2\cos^2(\theta_0-h_r)-v_r^2\cos^2 h_r(x_1^2+x_1\beta+r_0\,S\beta)+r_0^2v_r^2S\sin\theta_0\end{aligned}\quad\text{(A10)}$$

But, again from Lemma L2, we have $x_1^2 + \beta x_1 + r_0 \beta S = r_0^2$.

We simplify (A10) and we get

$$D = r_0^2 v_r^2 \cos^2(\theta_0 - h_r) - r_0^2 v_r^2 \cos^2 h_r + r_0^2 v_r^2 S \sin\theta_0$$
$$= r_0^2 v_r^2 \left[\cos^2(\theta_0 - h_r) - \cos^2 h_r + \sin(\theta_0 - 2h_r)\sin\theta_0 \right].$$

Using Lemma L1, we conclude that $D = 0$.

QED.

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Claude Jauffret born in France on March 29, 1957, received the diplôme d'Etudes Appronfondies in Applied Mathematics from Saint Charles University, Marseille, France, in 1981, the Diplôme d'Ingénieur from Ecole Nationale Supérieure d'Informatique et de Mathématiques Appliqués de Grenoble, Grenoble, France, in 1983, the title of « Docteur de l'Université » in 1993 and the « Habilitation à Diriger des Recherches » from the Université de Toulon et du Var, France. From Nov. 1983 to Nov. 1988, he worked on passive sonar systems, more precisely on target motion analysis at GERDSM, France. After a sabbatical year at the University of Connecticut (from Nov. 1988 to Dec. 1989) during which he worked of tracking problems in cluttered environment, he developed researches in tracking, data fusion, and extraction in GERDSM. Since Sept. 1996, he has been at the Université de Toulon where he teaches statistical signal processing. His current researches are about observability, estimation in nonlinear systems as they appear in tracking and TMA problems.

Annie-Claude Pérez born in France on November 10, 1965, received the Diplôme d'Etudes Appronfondies in Optics and Image Processing from the Université de Toulon, France, in 1988 and the title of « Docteur de l'Université » in 1991 from the Université de Toulon, France. Since Sept. 1994, she has been at the Université de Toulon where she teaches electronic systems. Her researches were focused signal processing applied to biomedical systems before turning them to Target Motion Analysis.

Denis Pillon, comes from a small French mountain called "Le Jura". He received the Diplôme d'Ingénieur from Ecole Supérieure de Physique et de Chimie Industrielle (ESPCI-Paris) in 1977 and the Diplôme d'Etudes Approfondies in probability theory from the Jussieu University, Paris in 1977. From 1978 to 1979 he worked in the area of operational research at the Centre InterArmées de

Recherche Opérationnelle (CIRO) agency of the Ministère de la Défense. Since 1980 he has been employed by Thomson Compagny (now Thales Compagny), France. From 1980 to 1981 he worked on sonar array processing. From 1982 to 1986, he was consultant at the *groupe d'études et de recherche en detection sous-marine* (GERDSM) of the Ministère de la Défense, for which he worked on sea trial analysis and TMA. From 1987 to 1989, he worked on tracking and track clustering, and in 1990 he was appointed group leader of signal and data processing laboratory at Thomson Sintra company, Sophia Antipolis, France. From 1992 to 1999, he was consultant at the DCN sonar laboratory. He worked on new sonar arrays and various submarine combat systems. Since 2000, he has been sonar expert at Thales Underwater Systems (Sophia-Antipolis). His current researches are about passive sonar performances estimation, submarine and torpedo array processing, and long term prospective studies for submarine warfare. He is now retired and keeps working on his favourite topics (sonar, tracking, TMA …).