Bearings-Only Target Motion Analysis: Observability when the Observer Maneuvers Smoothly

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***Abstract*—Observability in bearings-only target analysis (BOTMA) is studied when the target is in constant-velocity motion and the observer maneuvers gently (a constant turn motion or a constant acceleration motion). During our study, we proved that the rendezvous routes of the observer and the target play a key role in this analysis. We establish necessary and sufficient conditions of observability and we identify the virtual (or ghost) targets giving the same bearings when the system is not observable.**

***Index Terms*— Target motion analysis, tracking, bearings-only, observability, constant turn motion, constant acceleration motion, sonar, radar, electronic support measurement.**

I. INTRODUCTION

IT is well known that in bearings-only target motion analysis (BOTMA), under the classic assumption - the target is in constant velocity (CV) motion -, the observer must maneuver in order to render the trajectory of the target observable. Unfortunately, some maneuvers were proven to be ineffective, that is the trajectory of the target is still unobservable, although the maneuvers of the observer [6], [8], [9], [10], [2]. Because of their complex and non-intuitive mathematical expressions ([6] [10]), most of them seem difficult to be done in practice (a boat or a submarine cannot follow strictly a mathematical curve). Therefore, we can expect that simple maneuvers such as the constant turn (CT) motion or the constant acceleration (CA) motion are not ineffective maneuvers. For example, some authors considered in the past that a CT motion guaranteed implicitly observability (see [1] [4]). It turns out that observability has not been studied for these simple maneuvers. In this paper, we goal to analyze observability in BOTMA, when the observer is in CT motion, then we extend this analysis to constant acceleration (CA) motion, which is still a maneuver easy to do.

The paper is organized in five main sections:

In Section II, the notations are given and the observability problem posed. Section III is devoted to the types of kinematic we will study or encounter in this paper. We give in section IV two general results of BOTMA. The case of an observer in CT motion is studied in Section V. Finally, Section VI presents some results when the observer is in CA motion. The conclusion ends this paper.

This paper is linked with [7], in which we face the same problem, when the measurements are range.

II. PROBLEM STATEMENT AND NOTATIONS

A. Definitions and notations

Let a target (T) and an observer (O) moving in the same plane, given a Cartesian system. The target is in CV motion, but the observer maneuvers (that is, its velocity is not the same all along the scenario). The observation starts at time $t=0$ and finishes at time $t=T_f$. At time *t*, the position and velocity of the observer are respectively $P_O(t)=[x_O(t)\ y_O(t)]^\top$ and $V_O(t)=\frac{dP_O(t)}{dt}=[\dot{x}_O(t)\ \dot{y}_O(t)]^\top$. For the target, the notations are the same, except the subscript: $P_T(t)=[x_T(t)\ y_T(t)]^\top$, $V_T=\frac{dP_T(t)}{dt}=[\dot{x}_T\ \dot{y}_T]^\top$. In short, the respective motions of the observer and of the target are given by the vectors $X_O(t)=[x_O(t)\quad y_O(t)\quad \dot{x}_O(t)\quad \dot{y}_O(t)]^\top$ and $X_T(t)=[x_T(t)\quad y_T(t)\quad \dot{x}_T\quad \dot{y}_T]^\top$.

It is also convenient to define the relative motion of the target w.r.t the observer: $P_{OT}(t)=P_T(t)-P_O(t)=[x_{OT}(t)\ y_{OT}(t)]^\top$, $V_{OT}(t)=\frac{dP_{OT}(t)}{dt}=[\dot{x}_{OT}(t)\ \dot{y}_{OT}(t)]^\top$. Consequently, we define the vector $X_{OT}(t)=X_T(t)-X_O(t)=[x_{OT}(t)\quad y_{OT}(t)\quad \dot{x}_{OT}(t)\quad \dot{y}_{OT}(t)]^\top$. In the sequel, $X_T(0)$ will be denoted as X_T, which defines the target's trajectory, since $P_T(t)=P_T(0)+tV_T$. This vector will be named state vector. Similarly, $X_{OT}(0)$ will be denoted simply X_{OT}. We implicitly assume that $P_{OT}(t)\neq\vec{0}$, in $[0\quad T_f]$.

Vectors U and W are said collinear if a non-zero scalar α exists such that $\mathsf{U}=\alpha\mathsf{W}$. The angles are clockwise-positive. Subsequently, we will use the symbol $\angle$ to designate angles: for any pair of vectors U and W, $\angle(\mathsf{U},\mathsf{W})$ is the angle defined by the couple (U,W) referenced to U. When U is collinear to the northward direction, we will use $\angle\mathsf{W}$ only (for the bearing or heading).

The range and the bearing at time t are given respectively by

$r(t) = \| P_{OT}(t) \|$ and $\theta(t) = \angle P_{OT}(t)$.

Subsequently, we will use the following simplified notations: $\theta_0 = \theta(0)$, $r_0 = r(0)$, $v_r = \| V_{OT}(0) \|$, and $h_r = \angle V_{OT}(0)$ for the initial bearing, range, relative speed, and heading.

Figure 1 illustrates the notations for a typical scenario.

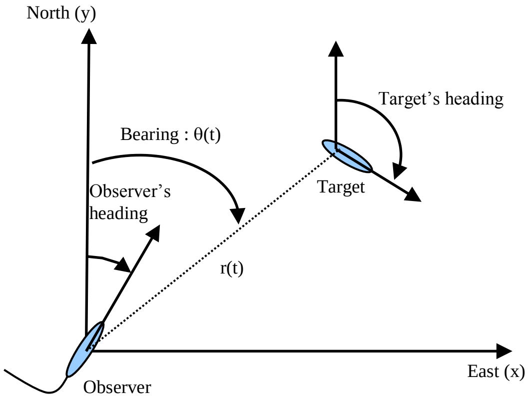

Figure 1. Typical scenario of TMA and associated notations

B. Observability notion

To emphasize the functional link between $\theta(t)$ and X_{OT}, we will denote $\theta(t)$ by $\theta(t, X_{OT})$.

We recall that the target's trajectory is declared observable in BOTMA if the following statement is true: $\forall t \in [0, T_f]$, $\theta(t, X) = \theta(t, X_{OT}) \Rightarrow X = X_{OT}$. Otherwise, the trajectory is unobservable: at least, one vector $X_{OG} = [x_{OG}(0) \quad y_{OG}(0) \quad \dot{x}_{OG} \quad \dot{y}_{OG}]^{\mathsf{T}}$ (defining a CV motion) different from X_{OT} exists such that $\theta(t, X_{OG}) = \theta(t, X_{OT}), \forall t \in [0, T_f]$.

The vector $X_G = X_{OG} + X_O$ defines the "virtual" trajectory of a "ghost-target", denoted G. For each ghost-target, we define similarly $P_G(0) = [x_G(0) \quad y_G(0)]^{\mathsf{T}}$, $V_G = [\dot{x}_G \ \dot{y}_G]^{\mathsf{T}}$,

$P_G(t) = P_G(0) + t V_G = [x_G(t) \ y_G(t)]^{\mathsf{T}}$, and

$X_G(t) = [x_G(t) \quad y_G(t) \quad \dot{x}_G \quad \dot{y}_G]^{\mathsf{T}}$, with the convention

$X_G = X_G(0)$.

Observability is a necessary (but not sufficient) condition to perform successful TMA. Observability analysis is based upon ideal situation: noiseless measurements and exactly known trajectory. Obviously, in real situation these hypotheses are not verified. The philosophy is to say: if in ideal situation estimating the trajectory of a target is impossible because of the presence of several solutions, in real situation it will be as well.

Observability analysis has two goals:

a) Give a necessary and sufficient condition to have unicity of the state vector X_{OT} (that is the trajectory of the target is observable),
b) When this trajectory is unobservable, characterize the set of X_{OG}.

III. Observer cinematic models

In this paper, we are concerned by two models of smooth observer motion: (i) the observer travels in an arc of a circle at constant speed, (ii) it has a constant acceleration vector.

A. CT motion

The observer travels in an arc of a circle whose center is a fixed point $P_C = \begin{bmatrix} x_C \\ y_C \end{bmatrix}$ and radius is $\rho > 0$. It has a constant turn rate $\omega \neq 0$ and an "initial phase" φ relative to north, at the beginning of its motion. So, the location of the observer is given by $P_O(t) = P_C + \rho \begin{bmatrix} \sin(\omega t + \varphi) \\ \cos(\omega t + \varphi) \end{bmatrix}$ (see Fig. 2). Note that its speed is constant. In order to simplify the coming calculation, we will assume that $P_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

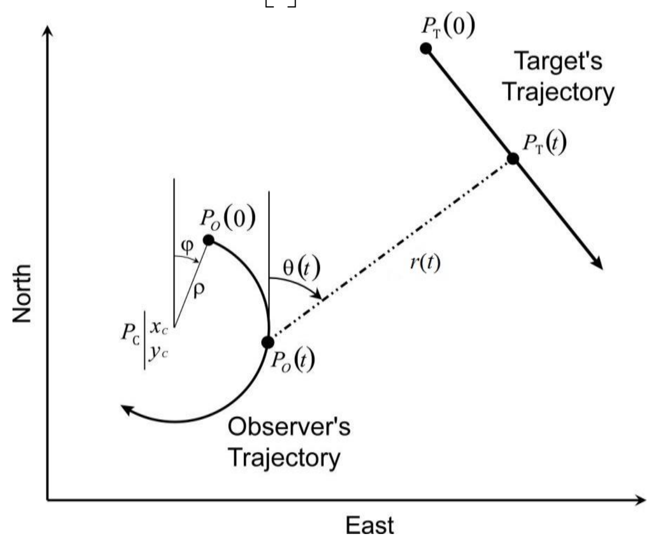

Figure 2. Typical scenario when the observer is traveling in an arc of a circle.

B. CA motion

The position of the observer at any time t is $P_O(t) = P_O(0) + t V_O(0) + \frac{t^2}{2}\Gamma$, where $\Gamma = [\gamma_x \quad \gamma_y]^{\mathsf{T}}$ is the acceleration vector. The relative position of the target with respect to the observer is

$$P_{OT}(t) = P_{OT}(0) + t V_{OT}(0) - \frac{t^2}{2}\Gamma \quad (1)$$

Without loss of generality, we will assume that $\gamma_x < 0$ and $\gamma_y = 0$. Indeed, a suitable rotation (depending on the vector Γ) of the entire scenario allows us to be in this case. This assumption will make observability analysis easier.

C. Rendezvous routes in CA motion

The observability criteria when the observer is in CA motion will be shown to be linked to the rendezvous (or "collision") route in Sections VI.

Definition: the rendezvous route
The target and the observer are said to be on a rendezvous route (RDVR), when they are in the same place at a time t_c.

Since $P_{OT}(t) \neq \vec{0}$, in $[0 \;\; T_f]$, t_c is not in $[0, T_f]$. Actually, these rendezvous instants are purely virtual: before $t=0$ and after T_f, O and T were and will be free to choose their own trajectories. Note that a CA motion is not the pursuit curve motion, which has not been yet studied in the TMA observability problem.

1) *The two types of RDVR*

Proposition 1: General properties of RDVR
If O and T are on an RDVR, then
- *Either* $P_{OT}(0)$, $V_{OT}(0)$ *and* Γ *are collinear,*
- *Or* $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear.*

Proof:
O and T collide at t_c if and only if

$$\begin{cases} x_{OT}(0) + t_c \dot{x}_{OT}(0) - \frac{1}{2} t_c^2 \gamma_x = 0 & (2) \\ y_{OT}(0) + t_c \dot{y}_{OT}(0) = 0 & (3) \end{cases}$$

If $y_{OT}(0)=0$, then (3) implies that $\dot{y}_{OT}(0)=0$. The vectors $P_{OT}(0)$, $V_{OT}(0)$ and Γ are collinear.

Else, $\dot{y}_{OT}(0) = -\frac{y_{OT}(0)}{t_c} \neq 0$. In other words, $P_{OT}(0)$ and Γ are noncollinear, and $V_{OT}(0)$ and Γ are noncollinear.

QED.

Definition: The two types of RDVR
The RDVRs are called rendezvous routes of type I (RDVR-I), when $P_{OT}(0)$, $V_{OT}(0)$, *and* Γ *are collinear. When* $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear as well, the RDVRs are called rendezvous routes of type II (RDVR-II).*

Note that for the RDVR-II, $P_{OT}(0)$ *and* $V_{OT}(0)$ can be collinear.

The converse of Proposition 1 is given in the following two propositions.

Proposition 2: Condition of RDVR-I
Assume that $P_{OT}(0)$, $V_{OT}(0)$ *and* Γ *are collinear; that is,* $P_{OT}(0) = \eta\Gamma$ *(with* $\eta \neq 0$*) and* $V_{OT}(0) = \lambda\Gamma$. *O and T are on an RDVR if and only if* $\lambda^2 \geq -2\eta$.

Proof:
O and T are on an RDVR, if and only if $x_{OT}(0) + t_c \dot{x}_{OT}(0) - \frac{1}{2} t_c^2 \gamma_x = 0$ (we do not have any equation with the y-component since $\gamma_y = 0$). This equality is equivalent to
$\left(\eta + \lambda t_c - \frac{1}{2} t_c^2\right) \gamma_x = 0$ or $t_c^2 - 2\lambda t_c - 2\eta = 0$.
Hence, O and T are on an RDVR if and only if the equation $t^2 - 2\lambda t - 2\eta = 0$ has one or two real roots (one of them is t_c), that is, if and only if the discriminant $\Delta = \lambda^2 + 2\eta$ is positive.

QED.

Remark: in this case, the bearings are piecewise equal to $\pm\frac{\pi}{2}$, and Proposition 2 remains valid up to a rotation, that is, for $\theta(t) = \text{constant}$ up to t_c. Note also that two rendezvous instants may exist (depending on Δ). In Fig. 3, an example of such a situation is given: The target starts at $[-4000 \quad 0]^T$ (m) with a velocity of $[20 \quad 2]^T$ (m/s); the observer starts at $[0 \quad 0]^T$ (m) with a velocity of $[10 \quad 2]^T$ (m/s) and its acceleration vector is $[-0.0416 \quad 0]^T$ (m/s^2).

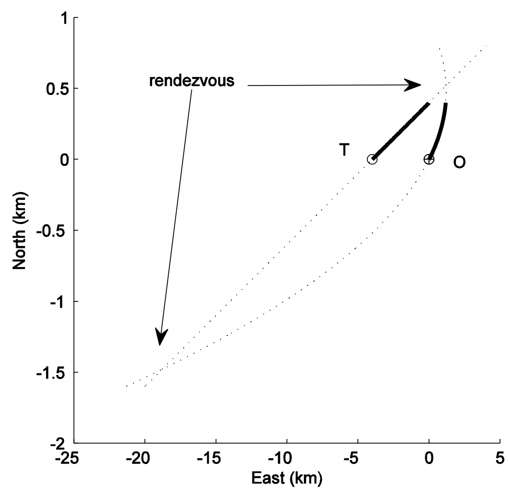

Figure 3. Example of case where the target and the observer have two RDVs.

Proposition 3: Condition of RDVR-II.
Assume that $P_{OT}(0)$ *and* Γ *are noncollinear, and* $V_{OT}(0)$ *and* Γ *are noncollinear as well.*
O and T are on a RDVR if and only if $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$.

Proof:

Equation (3) implies that $t_c = -\frac{y_{OT}(0)}{\dot{y}_{OT}(0)}$. Reporting this result in (2), we get $x_{OT}(0) - \frac{y_{OT}(0)}{\dot{y}_{OT}(0)}\dot{x}_{OT}(0) - \frac{1}{2}\frac{y_{OT}^2(0)}{\dot{y}_{OT}^2(0)}\gamma_x = 0$. We end up with $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$.

QED.

In the following proposition, we give a criterion based on the bearings, which allows us to know if we are on an RDVR-II or not.

2) Criterion of RDV-II

From the remark following Proposition 2, when the target and the observer are on an RDVR-I, the bearings are constant. In this case, the trajectory of the target is not observable in BOTMA. Conversely, if the bearings are constant, the two vehicles are not necessarily on an RDVR-I. Therefore, finding a criterion of RDVR-I is hopeless. It is impossible to decide if we are in logic of RDVR-I from bearings only.
In contrary, we can give a criterion of RDVR-II, based on bearings.

Proposition 4:
O and T are on RDVR-II if and only if $\tan\theta(t) = \mu_0 + \mu_1 t$, *with* $\mu_1 \neq 0$.

The respective values of μ_0 and μ_1 *are* $\frac{x_{OT}(0)}{y_{OT}(0)}$, *and* $\frac{y_{OT}(0)\dot{x}_{OT}(0) - \dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$.

Proof:
We have to prove that the equivalence $\tan\theta(t) = \mu_0 + \mu_1 t \iff \gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$

We have $\tan\theta(t) = \dfrac{x_{OT}(0) + t\,\dot{x}_{OT}(0) - \frac{1}{2}t^2\gamma_x}{y_{OT}(0) + t\,\dot{y}_{OT}(0)}$.

If $\tan\theta(t) = \mu_0 + \mu_1 t$ then
$$x_{OT}(0) + t\,\dot{x}_{OT}(0) - \frac{1}{2}t^2\gamma_x = \left[y_{OT}(0) + t\,\dot{y}_{OT}(0)\right]\left[\mu_0 + \mu_1 t\right]$$

As a consequence, $t_c = -\frac{y_{OT}(0)}{\dot{y}_{OT}(0)}$ is a root of $x_{OT}(0) + t\,\dot{x}_{OT}(0) - \frac{1}{2}t^2\gamma_x$. We get the equality $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$.

Conversely, if $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$, then we readily verify that $t_c = -\frac{y_{OT}(0)}{\dot{y}_{OT}(0)}$ is a root of $x_{OT}(0) + t\,\dot{x}_{OT}(0) - \frac{1}{2}t^2\gamma_x$. Hence, there are two reals numbers μ_0 and μ_1 such that
$$x_{OT}(0) + t\,\dot{x}_{OT}(0) - \frac{1}{2}t^2\gamma_x = \left[y_{OT}(0) + t\,\dot{y}_{OT}(0)\right]\left[\mu_0 + \mu_1 t\right].$$

QED.

To resume, if the bearings collected by the observer verify $\tan\theta(t) = \mu_0 + \mu_1 t$, then if $\mu_1 \neq 0$, the observer and the target are on an RDVR-II; else, they can be on an RDVR-I or not.

In practice, it is well known by sailors that when the observer is itself in CV motion and the bearings are constant, the observer and the target can be on a collision route. To avoid a collision, the observer must maneuver. However, if it accelerates and if the bearing tangent is linear, it may still remain on a rendezvous route, and the collision will occur.

IV. Two General Results About Observability in BOTMA

We give hereafter two general results, which will be necessary to prove observability when the trajectory of the target is a combination of CA and CV motions (see Proposition 11).

Proposition 5: Observability equivalence between two observers in BOTMA
Let there be two observers measuring the same bearings. If the target is observable from one, it will be observable from the other (or equivalently, if it is unobservable from one, it will be unobservable from the other).

Proof:
We recall that BOTMA has a linear version, whatever the trajectory of the ownship. Indeed, the noise-free measurement equation $\theta(t) = \tan^{-1}\left[\frac{x_T(0) + t\,\dot{x}_T - x_O(t)}{y_T(0) + t\,\dot{y}_T - y_O(t)}\right]$ can be transformed into the linear equation [3]
$$\left[\cos\theta(t) \quad -\sin\theta(t) \quad t\cos\theta(t) \quad -t\sin\theta(t)\right]X_T$$
$$= x_O(t)\cos\theta(t) - y_O(t)\sin\theta(t),\ \forall\ t \in \{t_1, \cdots, t_N\}$$
or, in short, $\mathbf{A}(\theta)X_T = Z$, where the k-th line of $\mathbf{A}(\theta)$ is $\left[\cos\theta(t_k) \quad -\sin\theta(t_k) \quad t_k\cos\theta(t_k) \quad -t_k\sin\theta(t_k)\right]$, and the k-th element of Z is $x_O(t_k)\cos\theta(t_k) - y_O(t_k)\sin\theta(t_k)$; see [3, 11, 16]. The observability is hence brought by the set $\{\theta(t_1), \cdots, \theta(t_N)\}$, which means that if two observers collect the same set $\{\theta(t_1), \cdots, \theta(t_N)\}$, if the target is observable from one, it will be observable from the second one. In this case, the state vector is computed by $X_T = \left[\mathbf{A}^\top(\theta)\,\mathbf{A}(\theta)\right]^{-1}Z$.

QED.

Proposition 6: Observability equivalence for time-reversed

bearings
Let there be two observers #1 and #2. Observer #2 measures the same bearings as observer #1, but in the inverse temporal order, that is $\theta_{(2)}(t_k)=\theta_{(1)}(t_N-t_k)$, where $\theta_{(i)}(t_k)$ is the bearing measured at time t_k by observer #i. If the target detected by observer #1 is observable, then the target detected by observer #2 will be, and the converse.

Proof:
Let us define the matrix $\mathbf{A}(\theta_{(i)})$ whose k-th line is $\left[\cos\theta_{(i)}(t_k) \quad -\sin\theta_{(i)}(t_k) \quad t_k\cos\theta_{(i)}(t_k) \quad -t_k\sin\theta_{(i)}(t_k)\right]$. Let us prove that $\text{Rank}\,\{\mathbf{A}(\theta_{(1)})\}=\text{Rank}\,\{\mathbf{A}(\theta_{(2)})\}$. We note that the k-th line of $\mathbf{A}(\theta_{(2)})$ is

$$\left[\cos\theta_{(2)}(t_k) \quad -\sin\theta_{(2)}(t_k) \quad t_k\cos\theta_{(2)}(t_k) \quad -t_k\sin\theta_{(2)}(t_k)\right]$$
$$=\left[\cos\theta_{(1)}(t_N-t_k) \quad -\sin\theta_{(1)}(t_N-t_k) \quad t_k\cos\theta_{(1)}(t_N-t_k) \quad -t_k\sin\theta_{(1)}(t_N-t_k)\right]$$

We construct a third matrix denoted $\tilde{\mathbf{A}}$ by permutation of the lines of $\mathbf{A}(\theta_{(2)})$, that is, the first line of $\tilde{\mathbf{A}}$ is the N-th line of $\mathbf{A}(\theta_{(2)})$, the second line of $\mathbf{A}(\theta_{(2)})$ is the (N – 1)-th of $\mathbf{A}(\theta_{(2)})$, and so on. In other words, we flip the matrix in the up/down direction. Obviously, $\text{Rank}\,\{\tilde{\mathbf{A}}\}=\text{Rank}\,\{\mathbf{A}(\theta_{(2)})\}$. We note that the first two columns of $\tilde{\mathbf{A}}$ are the first two columns of $\mathbf{A}(\theta_{(1)})$. The third (resp. fourth) column of $\tilde{\mathbf{A}}$ is t_N multiplied by the first (resp., second) column of t_N, minus the third (resp., fourth) column of $\mathbf{A}(\theta_{(1)})$. Hence, $\text{Rank}\,\{\tilde{\mathbf{A}}\}=\text{Rank}\,\{\mathbf{A}(\theta_{(1)})\}$. Consequently, $\text{Rank}\,\{\mathbf{A}(\theta_{(2)})\}=\text{Rank}\,\{\mathbf{A}(\theta_{(1)})\}$. We readily deduce that $\text{Rank}\{\mathbf{A}^{\mathsf{T}}(\theta_{(1)})\,\mathbf{A}(\theta_{(1)})\}=\text{Rank}\{\mathbf{A}^{\mathsf{T}}(\theta_{(2)})\,\mathbf{A}(\theta_{(2)})\}$.
QED.

Note that these properties cannot be extended for any measurements such as frequency measurements, since they are not time-reversible.

V. Observer in CT motion

The CT motion was defined in Section III A. We propose the following result when the noise-free bearings are continuously available during $[0, T_f]$.

Proposition 7: Observability in BOTMA for CT motion
If the observer is traveling along an arc of a circle, then any target moving with a constant velocity is observable in BOTMA.

Proof:
Suppose that a ghost (G) moving in CV motion is detected in the same bearings as the target for any $t\in[0, T_f]$.

The equality $\theta(t)=\angle P_{OT}(t)=\angle P_{OG}(t)$ is equivalent to $P_{OG}(t)=k(t)P_{OT}(t)$ for certain $k(t)>0$

$$\Leftrightarrow P_G(t)-P_O(t)=k(t)[P_T(t)-P_O(t)], \quad \forall\, t\in[0, T_f]$$
$$\Leftrightarrow P_G(t)=k(t)P_T(t)+[k(t)-1]P_O(t)$$
$$\Leftrightarrow P_G(0)+tV_G=k(t)[P_T(0)+tV_T]+\rho[k(t)-1]\begin{bmatrix}\sin(\omega t+\varphi)\\ \cos(\omega t+\varphi)\end{bmatrix}$$
$$\Leftrightarrow k(t)=1.$$

In other words, no such ghost exists: the trajectory of the target is hence observable.

QED.

Note that if the observer's trajectory contains at least one arc of a circle, then the trajectory of any target having a constant velocity is observable in BOTMA.

VI. Observer in CA motion

We have now the tools to give conditions observability criterion based on bearings only.

Proposition 8:
Assume that the observer is in CA motion.
The target's trajectory is unobservable if and only if $\tan\theta(t)=\mu_0+\mu_1 t$.
If $\mu_1\neq 0$, *the target and the observer are on a RDVR-II and the ghosts are on a RDVR-II with the observer. Their trajectories are defined by* $X_G(0)=X_T(0)+\alpha\,\Xi$, *where* α *is a scalar and* Ξ *is a vector of the null space of the matrix*
$$\begin{bmatrix} y_{OT}(0) & -x_{OT}(0) & 0 & 0\\ \dot{y}_{OT}(0) & -\dot{x}_{OT}(0) & y_{OT}(0) & 0\\ 0 & -\dfrac{\gamma_x}{2} & -\dot{y}_{OT}(0) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$
If $\mu_1=0$, the trajectories of the ghosts are defined by $X_G(0)=X_T(0)+\,\boldsymbol{\Xi}'$, *with* $\boldsymbol{\Xi}'=[\xi_1' \quad 0 \quad \xi_2' \quad 0]^{\mathsf{T}}$.

Proof:
We have to identify the solutions of the equation $\theta(t, X)=\theta(t, X_{OT})$. Since the implication $\{\theta(t, X)=\theta(t, X_{OT}) \;\Rightarrow\; \tan\theta(t, X)=\tan\theta(t, X_{OT})\}$ holds, we concentrate our effort on the equation $\tan\theta(t, X)=\tan\theta(t, X_{OT})$. We define the components of X by $[x \quad y \quad \dot{x} \quad \dot{y}]^{\mathsf{T}}$.

Two cases must be studied:

Case (1): $y_{OT}(0)\neq 0$ or $\dot{y}_{OT}(0)\neq 0$.

$$\Rightarrow \quad \frac{x+t\,\dot{x}-\frac{t^2}{2}\gamma_x}{y+t\,\dot{y}} = \frac{x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{t^2}{2}\gamma_x}{y_{OT}(0)+t\,\dot{y}_{OT}(0)},\ \forall\, t \in [0, T_f]\ .$$

$$\Leftrightarrow \left(x+t\,\dot{x}-\frac{t^2}{2}\gamma_x\right)\left[y_{OT}(0)+t\,\dot{y}_{OT}(0)\right] = \left[x_{OT}(0)+t\,\dot{x}_{OT}(0)-\frac{t^2}{2}\gamma_x\right](y+t\,\dot{y}),\ \forall\, t \in [0, T_f]\ .$$

After reordering the terms of this equation, we get

$$\begin{cases} x_{OT}(0)y = y_{OT}(0)x \\ x_{OT}(0)\dot{y}+\dot{x}_{OT}(0)y = \dot{y}_{OT}(0)x + y_{OT}(0)\dot{x} \\ \dot{x}_{OT}(0)\dot{y}-\frac{\gamma_x}{2}y = \dot{y}_{OT}(0)\dot{x}-\frac{\gamma_x}{2}y_{OT}(0) \\ \dot{y} = \dot{y}_{OT}(0) \end{cases}$$

We end up with the system

$$\mathbf{M}\,X = B \qquad (\Sigma)$$

with
$$\mathbf{M} = \begin{bmatrix} y_{OT}(0) & -x_{OT}(0) & 0 & 0 \\ \dot{y}_{OT}(0) & -\dot{x}_{OT}(0) & y_{OT}(0) & 0 \\ 0 & -\frac{\gamma_x}{2} & -\dot{y}_{OT}(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
and
$$B = \begin{bmatrix} 0 \\ x_{OT}(0)\dot{y}_{OT}(0) \\ -\dot{x}_{OT}(0)\dot{y}_{OT}(0)-\frac{\gamma_x}{2}y_{OT}(0) \\ \dot{y}_{OT}(0) \end{bmatrix}$$

We note that the vector X_{OT} is a solution of the above equation.

The vector X_{OT} is the unique solution of (Σ) if and only if $\det(\mathbf{M}) \neq 0$, in other words, the trajectory of the target is observable. And conversely, the trajectory of the target is not observable if and only if $\det(\mathbf{M}) = 0$. Consequently, the discussion is about $\det(\mathbf{M})$.

We readily get

$$\det(\mathbf{M}) = -x_{OT}(0)\dot{y}_{OT}^2(0) + y_{OT}(0)x_{OT}(0)\dot{y}_{OT}(0) + \frac{1}{2}y_{OT}^2(0)\gamma_x\,.$$

If $\det(\mathbf{M}) = 0$, we have to discriminate the subcase where $y_{OT}(0) \neq 0$ from the subcase $y_{OT}(0) = 0$.

If $y_{OT}(0) = 0$, then $x_{OT}(0) = 0$. The case $x_{OT}(0) = 0$ means that the target and the observer are co-localized at the initial time. This does not respect the assumptions given in II A. This case is hence discarded.

If $y_{OT}(0) \neq 0$, then $\gamma_x = 2\frac{\dot{y}_{OT}(0)}{y_{OT}^2(0)}\left[x_{OT}(0)\dot{y}_{OT}(0) - \dot{x}_{OT}(0)y_{OT}(0)\right]$; that is, the observer and the target are on an RDVR-II (cf. Proposition 3). Since the acceleration γ_x is not equal to zero, $\dot{y}_{OT}(0)$ cannot be equal to zero. As the consequence, the first two columns of $\mathbf{M}$ are not collinear. We remark also that the fourth column of the matrix $\mathbf{M}$ cannot be simultaneously collinear with another one, so the rank of $\mathbf{M}$ is equal to 3.

The set of solutions of (Σ) is the set of the vectors defined by $X_{OG} = X_{OT} + \alpha\,\Xi$, where α is a scalar and $\Xi = \begin{bmatrix}\xi_1 & \xi_2 & \xi_3 & 0\end{bmatrix}^{\mathrm{T}}$ is a nonzero vector of the null space of $\mathbf{M}$. We verify that $\theta(t)$ has a special form: $\theta(t) = \tan^{-1}(\mu_0 + \mu_1 t)$, with $\mu_0 = \frac{x_{OT}(0)}{y_{OT}(0)}$ and $\mu_1 = \frac{y_{OT}(0)\dot{x}_{OT}(0) - \dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$.

Note that $\frac{x_{OG}(0)+t\,\dot{x}_{OG}(0)-\frac{t^2}{2}\gamma_x}{y_{OG}(0)+t\,\dot{y}_{OG}(0)} = \mu_0 + \mu_1 t$ is equivalent to $x_{OG}(0)+t\,\dot{x}_{OG}(0)-\frac{t^2}{2}\gamma_x = (\mu_0+\mu_1 t)(y_{OG}(0)+t\,\dot{y}_{OG}(0))$.

Hence, at time $t_{G,c} = -\frac{y_{OG}(0)}{\dot{y}_{OG}(0)} = -\frac{y_{OG}(0)}{\dot{y}_{OT}(0)}$ (which depends on the relative coordinate $y_{OG}(0)$ of the considered ghost), we get $x_{OG}(t_{G,c}) = 0$ and $y_{OG}(t_{G,c}) = 0$. As a consequence, all the ghosts and the observer are on an RDVR-II (but at different times of rendezvous).

Case (2): $y_{OT}(0) = 0$ and $\dot{y}_{OT}(0) = 0$; that is, $P_{OT}(0), V_{OT}(0)$ and Γ are collinear. Note that O and T are not necessarily on an RDVR-I.
Then $\theta(t,\, X) = \theta(t,\, X_{OT}) = \pm\frac{\pi}{2} \Leftrightarrow y + t\,\dot{y} = 0$; that is, $y = \dot{y} = 0$.
The bearing rate is zero and the set of solutions is the line of sight of the target: any ghost traveling in this line (the X-axis) is detected in the same (constant) bearing $\left(\pm\frac{\pi}{2}\right)$ as the target of interest. The target's trajectory is not observable.

QED.

We end up with an observability criterion based on bearings only.

Proposition 9: Observability criterion for CA motion
Assume that the observer is in CA motion.
The target's trajectory is observable if and only if $\tan\theta(t) \neq \mu_0 + \mu_1 t$ *or equivalently, if and only if O and T are not on an RDVR-II and the bearings are not constant.*

Proposition 10

The trajectory of the target is not observable if, and only if a virtual observer, located at $P_O(0)$ at time $t=0$, in CV motion, continuously collects the same bearings as the observer.

If the bearings are constant, the observer, the target and the virtual observer travel in the same line.

*If O and T are on an RDVR-II, i*ts velocity is

$$V_E=\begin{bmatrix}\dot{x}_O(0)+\dot{y}_{OT}(0)\dfrac{x_{OT}(0)}{y_{OT}(0)}\\ \dot{y}_T\end{bmatrix}.$$

Proof:

We denote in this proof the initial position and the velocity of a virtual observer in CV motion by $P_E(0)$ and V_E, respectively.

If a virtual observer, located at $P_O(0)$ at time $t=0$, in CV motion, continuously collecting the same bearings as the observer exists, the trajectory of the target is not observable from Proposition 5.

Conversely, if the trajectory of the target is not observable, then from proposition 9, the bearings are constant or the target and the observer are on a RDVR II.

If the bearings are constant, then $y_O(0)=\dot{y}_O(0)=0$. The virtual observer (in CV motion) such that $P_E(0)=P_O(0)$ and $V_E=V_O(0)$ collects the same bearings as O.

If the target and the observer are on a RDVR II, let us construct a virtual observer E in CV motion: Its initial position and velocity are denoted $P_E(0)=\begin{bmatrix}x_O(0) & y_O(0)\end{bmatrix}^{\mathsf{T}}$ and $V_E=\begin{bmatrix}\dot{x}_E & \dot{y}_E\end{bmatrix}^{\mathsf{T}}$. The equality $\theta(t)=\tan^{-1}\left(\dfrac{x_{OT}(0)+t\,\dot{x}_{ET}}{y_{OT}(0)+t\,\dot{y}_{ET}}\right)$ implies that $\dfrac{x_{OT}(0)+t\,\dot{x}_{ET}}{y_{OT}(0)+t\,\dot{y}_{ET}}=\mu_0+\mu_1 t$. Consequently $\dot{y}_{ET}=0$, and $\dfrac{\dot{x}_{ET}}{y_{OT}(0)}=\mu_1$. Since $\mu_1=\dfrac{y_{OT}(0)\dot{x}_{OT}(0)-\dot{y}_{OT}(0)x_{OT}(0)}{y_{OT}^2(0)}$ (see Proposition 4), we get $\dot{x}_{ET}=\dot{x}_{OT}(0)-\dot{y}_{OT}(0)\dfrac{x_{OT}(0)}{y_{OT}(0)}$. We end up with $V_E=\begin{bmatrix}\dot{x}_O(0)+\dot{y}_{OT}(0)\dfrac{x_{OT}(0)}{y_{OT}(0)}\\ \dot{y}_T\end{bmatrix}$. Note that (i) this virtual observer is unique (ii) $V_E\neq V_O(0)$, otherwise $\dot{y}_{OT}(0)=0$ which implies that $\gamma_x=0$ from Proposition 3.

QED.

Recall that this notion of virtual observer was initially introduced in [6] for any ineffective maneuver: for such a maneuver, we have $\tan\theta(t)=\dfrac{\mu_0+\mu_1 t}{1+\mu_3 t}$. We prove here that $\mu_3=0$ if the ineffective maneuver corresponds to an RDVR-II.

Hereafter, we propose an example of a scenario where the observer has a higher order dynamic than the target and the target's trajectory still remains unobservable. We have chosen a scenario with an RDVR-II (hence satisfying Proposition 3): $P_O(0)=[\,0 \quad 0\,]^{\mathsf{T}}$, $V_O(0)=[\,10 \quad 2\,]^{\mathsf{T}}$ (m/s), $P_T(0)=[\,3000 \quad 4000\,]^{\mathsf{T}}$ (m), and $V_T=[\,-6 \quad -4\,]^{\mathsf{T}}$ (m/s) with $\gamma_x=-0.0345$ m/s^2. The duration is 6 minutes. In this case, $\tan\theta(t)=\mu_0+\mu_1 t$, with $\mu_0=3/4$ and $\mu_1=-0.0029$, which verifies Proposition 4.

Figure 4 depicts the maneuvering observer together with the target (thick lines) and four ghosts (thin lines). Moreover, the trajectory of the virtual observer E is plotted. Four lines of sight are given.

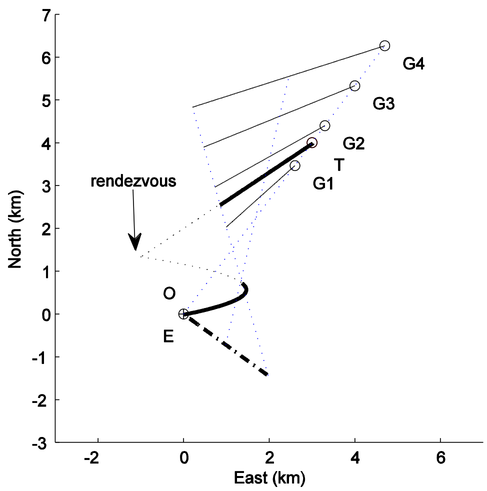

Figure 4. Non-observable trajectory in BOTMA, the target, some ghosts, and the bearing-equivalent-non-maneuvering observer.

Extension: Observer in CV and then CA motion and the converse:

In practice, an observer decides to maneuver after a phase of CV motion. Therefore, it is worth studying observability when the trajectory of the observer is composed of a CV motion, then of a CA motion. The velocity of the observer at the very beginning of the CA motion is equal to the velocity during the first leg.

Of course, observability must be studied when during the CA motion the trajectory of the target is not observable.

The case where the bearings are constant during the CA motion can be straightaway discarded since during the first leg, the bearings are the same than during the CA motion (and are constant).

Therefore, we will assume that O and T are on a RDVR-II during the CA motion, that is $\tan\theta(t)=\mu_0+\mu_1 t$.

Proposition 11: Observability criterion in BOTMA for CA-CV and CV-CA motions

Assume that the observer is in CV motion and then in CA motion or the converse. The target's trajectory is observable in BOTMA if the bearings are not constant during the CA motion.

Proof:

We have only to consider the case where O and T are on an

RDVR-II during the CA motion. During this phase, following Proposition 10, the observer collects the same bearings as the non-maneuvering observer E. Hence, the observer collects the bearings acquired by another observer whose trajectory is composed of two legs at CV: the first one is defined by $V_O(0)$, and the second one by V_E. In short, the observer acquires the same bearings as the ones collected by a leg-by-leg maneuvering observer. Since the bearings are not constant, the target's trajectory is observable from the virtual observer E (see [8]). Proposition 5 completes the proof.

QED

Remark: Proposition 6 allows us to extend Proposition 11 when the observer is first in CA motion and then in CV motion.

VII. Conclusion

In this paper, observability of a target in constant velocity motion from a smoothly maneuvering observer (constant turn motion and constant acceleration motion) has been conducted.

When the observer is in CT motion (see for example [1] and [4]), observability is always guaranteed. If the displacement of the observer contains at least an arc of a circle, this conclusion remains valid.

When the observer is in CA motion, observability is guaranteed if and only if there is no μ_0 and μ_1 such that the tangent of the bearings at time *t* is equal to $\mu_0 + \mu_1 t$.

In any other case, that is the tangent of the bearings at time t is equal to $\mu_0 + \mu_1 t$ for some μ_0 and μ_1, its trajectory is not observable. This proves that even if the observer kinematic is of an order greater than the one of the target, observability is not guaranteed (see eq (41) in [6]). This is not in contradiction with [5], whose authors established a necessary (but non-sufficient) observability condition. We proved that if μ_1 is non null, then the observer and the target are on a rendezvous route.

We characterized the set of ghost-targets, which is uncountable.

We extended our analysis when the observer's trajectory is composed of a CA motion followed by a CV motion (and inversely): arguing fundamental properties, we proved that the target is observable in this case.

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