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INFLUENCE OF A NONLINEAR DEGENERATE DIFFUSION ON AN ADVECTION-DIFFUSION EQUATION IN A DIFFUSE INTERFACE FRAMEWORK *

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Abstract. This work is motivated by the modelling a liquid-vapour flows with phase transition describing the evolution of the coolant within an heat exchanger (e.g. the core of a Pressurized Water Reactor). We investigate an advection-diffusion equation with a degenerate and nonlinear diffusion coefficient. The degeneracy corresponds to a liquid-vapor mixture in the original model whereas the diffusion coefficient is non-degenerate in the pure phase cases. We focus on the influence of the diffusion coefficient on a simple 1D configuration for which some analytical computations can be done, leading to a surprising behavior of the phase transition with respect to the diffusion.

Keywords: phase change, Stefan problem, nonlinear degenerate diffusion.

1. Context

A heat exchanger is a device that facilitates the process of heat exchange between two fluids that are at different temperatures. Heat exchangers are used in many engineering applications, such as refrigeration, heating and air-conditioning systems, power plants, chemical processing systems, automobile radiators, and waste heat recovery units. Nuclear core of a pressurized water reactor is an example of heat exchangers.

In some applications, such as the flow in nuclear reactor cores, convection is characterized by a low Mach number, where the convective velocities are much slower than the speed of sound in the fluid. This has promoted the development of so-called low Mach number models, which filter the sound waves. In the context of pressurized water reactor cores, an asymptotic low Mach number model, called LMNC (for Low Mach Nuclear Core), has been derived and investigated in a series of papers [1, 2, 4, 5, 7–9]. The model was derived through an asymptotic expansion performed in the compressible Navier-Stokes equations with an energy source term. It consists of a transport equation upon a thermodynamic variable (here the total enthalpy), of a divergence constraint upon the velocity (with a nonlinear coupling source term which underlines the dilatation property of the flow) and of the momentum equation. The fluid is described by a single equation of state taking into account the phase transition by supposing that, when both vapour and liquid phases are present, they have the same pressure, temperature and chemical potential. The equation of state is piecewise defined with respect to the enthalpy identifying a liquid phase, a gas phase and a mixture of them. The thermal diffusion degenerates in the mixture region.

In the present paper we are interested in studying the influence of the thermal diffusion.

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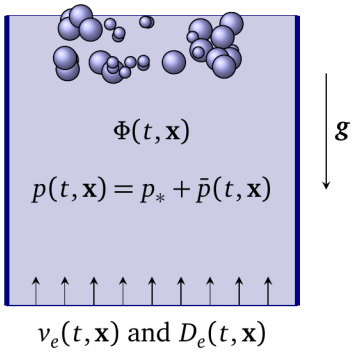


FIGURE 1. The heat exchanger configuration

1.1. **A heat exchanger model**

For some bounded domain $\Omega \subset \mathbb{R}^3$, the Navier-Stokes-Fourier system with source terms in a 3D conservative formulation reads

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho \boldsymbol{g} + \nabla \cdot \sigma(\mathbf{u}), \\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\omega \nabla T) + \sigma(\mathbf{u}) : \nabla \mathbf{u} + \partial_t p + \mathbf{u} \cdot \nabla p. \end{cases} \tag{1}$$

The unknows are the velocity field $(t, \mathbf{x}) \mapsto \mathbf{u}(t, \mathbf{x})$, the total enthalpy $(t, \mathbf{x}) \mapsto h(t, \mathbf{x})$ and the pressure $(t, \mathbf{x}) \mapsto p(t, \mathbf{x})$. The specific density ρ and the temperature T are related to the enthalpy h and the pressure p through an equation of state. The heat conductivity ω, assumed to be constant and isotropic for each phase, is also related to the enthalpy h and the pressure p. The power density $(t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x}) \geq 0$ is a given function of time and space modelling the heating of the coolant fluid (*e.g.* due to the fission reactions in the nuclear core). Finally, $\boldsymbol{g}$ is the gravity field and $\sigma(\mathbf{u})$ models viscous effects.

In a low Mach number regime (*i.e.* when the speed of fluid is much smaller than the speed of sound), an asymptotic expansions with respect to the Mach number can be applied to the compressible model, and this leads to a simpler model where the acoustic waves are filtered out. The "original" pressure p is decomposed in two pressure fields $p(t, \mathbf{x}) = p_* + \bar{p}(t, \mathbf{x})$, where p_* is the reference state pressure (that is an average pressure constant in time and space) and $\bar{p}$ is the perturbational pressure (often called "dynamic pressure"). For low Mach number flows, an asymptotic analysis shows that $\bar{p}(t, x)/p(t, x) = O(\mathcal{M}^2)$, where $\mathcal{M}$ is the Mach number. We can thus approximate $p(t, x)$ by p_* in the computation of thermodynamics quantities and we obtain the so-called LMNC model [1, 4, 8], that can be written as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \bar{p} = \rho \boldsymbol{g} + \nabla \cdot \sigma(\mathbf{u}), \\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\omega \nabla T) + \sigma(\mathbf{u}) : \nabla \mathbf{u}. \end{cases} \tag{2}$$

In this system the unknows are the velocity field $\mathbf{u}$, the enthalpy h and the perturbational pressure $(t, \mathbf{x}) \mapsto \bar{p}(t, \mathbf{x})$. The specific density ρ and the temperature T are linked, by an equation of state, to the enthalpy h and to the *constant* given thermodynamic pressure p_*.

In the following the power density Φ is chosen constant and positive. We also neglect the viscous terms.

1.2. **Diffusion term with phase transition**

The fluid can be in liquid phase ℓ, vapour phase g or a mixture of them m.

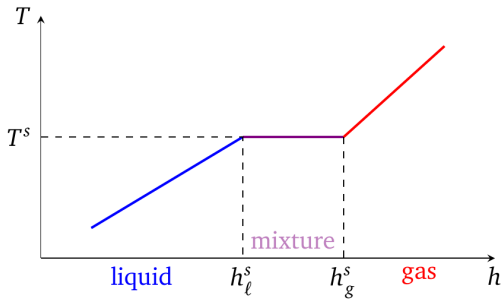


FIGURE 2. Temperature T as a function of the enthalpy h at a constant pressure p_*

Pure phase $\kappa \in \{\ell, g\}$: We consider each phase κ as a compressible fluid characterized by its thermodynamic properties, *i.e.* each phase is governed by a given (complete) equation of state (EoS), so that we can define the temperature $(h,p) \mapsto T_\kappa(h,p)$ as function of the enthalpy h and the pressure p. We can also define $(T,p) \mapsto h_\kappa(T,p)$.

Mixture $\kappa = m$: The mixture is supposed at saturation: when phases coexist they have the same pressure p, the same temperature T and the same chemical potential μ. Let us denote by $T^s(p)$ the solution of the equation $\mu_\ell(T,p) = \mu_g(T,p)$ (the so called *temperature at saturation*). We can then define $h^s_\kappa(p) \stackrel{\text{def}}{=} h_\kappa(T^s(p),p)$ the enthalpy of the phase κ at saturation.

Global equation of state: At any given pressure p, the fluid is in liquid phase if $h \le h^s_\ell(p)$, in vapor phase if $h \ge h^s_g(p)$ and a mixture at saturation if $h^s_\ell(p) < h < h^s_g(p)$. Thus the global EoS is piecewise defined w.r.t. $h^s_\kappa(p)$. In particular for the temperature we have (see Figure 2)

$$(h,p) \mapsto T(h,p) = \begin{cases} T_\ell(h,p), & \text{if } h \le h^s_\ell(p), \\ T_m(h,p) = T^s(p), & \text{if } h^s_\ell(p) < h < h^s_g(p), \\ T_g(h,p), & \text{if } h \ge h^s_g(p). \end{cases}$$

Since the temperature is a function of the enthalpy h and the (thermodynamic) pressure p, the gradient of the temperature can be written as

$$\nabla T(h) = \left.\frac{\partial T}{\partial h}\right|_p \nabla h + \left.\frac{\partial T}{\partial p}\right|_h \nabla p.$$

In the LMNC model, the thermodynamic quantities are evaluated at $p = p_*$, which is constant. Thus, from now letting drop the dependency upon p_*, the diffusion term can be written as

$$\omega(h)\nabla T(h) = \begin{cases} \lambda_\ell \nabla h, & \text{if } h \le h^s_\ell, \\ 0, & \text{if } h^s_\ell(p_*) < h < h^s_g, \\ \lambda_g \nabla h, & \text{if } h \ge h^s_g, \end{cases}$$

where $\lambda_\kappa \stackrel{\text{def}}{=} \omega_\kappa / c_{p,\kappa}$ for $\kappa = \ell$ or g with ω_κ the heat conductivity (assumed to be constant and isotropic for each pure phase κ) and $c_{p,\kappa} \stackrel{\text{def}}{=} \left.\frac{\partial h}{\partial T}\right|_p$ the isobar heat capacity of the pure phase κ. We note that if the open domain of the mixture is non-empty, a zero diffusion coefficient can be set in this region.

Let us denote

$$\lambda(h) \stackrel{\text{def}}{=} \begin{cases} \lambda_\ell & \text{if } h \le h^s_\ell, \\ \lambda_m & \text{if } h^s_\ell < h < h^s_g, \\ \lambda_g & \text{if } h \ge h^s_g. \end{cases} \tag{3}$$

By defining $\lambda_m = 0$ and by virtue of (2), with (3), the obtained equation on enthalpy comes down to

$$\partial_t(\rho h) - \nabla \cdot (\lambda(h)\nabla h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \sigma(\mathbf{u}) : \nabla \mathbf{u}.$$

It differs strongly on the enthalpy equation written in [13] where the transition liquid-gas is forced at a sharp interface by the addition of a singular source term and a non-degeneracy of the thermal diffusion. The generality of the model proposed here is that the liquid-gas transition can also take place through a sharp interface or through a mixing region (in a diffuse interface framework). In the following, we will see that both can occur and when the liquid-gas transition is a sharp interface, our enthalpy solution coincides with that of the Stefan models written in temperature [12] or enthalpy [13]. This is the main point of this paper and an analytic solution, derived in the 1D steady-state case in the following section, shows this richness of the phase transition.

2. Steady-state model

In this section, we focus on the steady-state solution, *i.e.* we seek for h, $\mathbf{u}$ and $\bar{p}$ solution in $\mathbb{R}^3_+$ of

$$\begin{cases} \nabla \cdot (\rho \mathbf{u}) = 0, \\ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \bar{p} = \rho \boldsymbol{g}, \\ \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\lambda(h) \nabla h). \end{cases}$$

In the unidimensional case, the first and last equations can be written as

$$\begin{cases} \partial_y(\rho v) = 0, \\ \partial_y(\rho h v) = \Phi + \partial_y(\lambda(h)\partial_y h) \end{cases} \quad \text{in } [0, +\infty) \tag{4}$$

with ρ a function of h. To model the configuration described in the Figure 1, the system is closed by the Dirichlet boundary condition corresponding to an inlet condition: the inlet enthalpy $h(0) = h_e < h_\ell^s$ and the inlet flow rate $(\rho v)(0) = D_e$. Then the flow rate is constant on the domain:

$$(\rho v)(y) = D_e \text{ for all } y \in \mathbb{R}^+. \tag{5}$$

In order to focus on a physically relevant solution, we assume that $\lim_{y\to\infty} \partial_y(\lambda(h)\partial_y h) = 0$ so that, thanks to (4),

$$\lim_{y\to\infty} \partial_y h(y) = \frac{\Phi}{D_e}.$$

We are thus interested in $y \mapsto h(y)$ solution of the following non-linear strongly degenerate convection-diffusion problem:

$$D_e h' - (\lambda(h)h')' = \Phi \quad \text{in } \mathbb{R}^+ \tag{6}$$

with

- Φ and D_e two strictly positive constants,
- $h \mapsto \lambda(h)$ the degenerate diffusion defined in (3),
- the two boundary conditions

$$\begin{cases} h(0) = h_e < h_\ell^s & \text{[inlet Dirichlet boundary condition]}, \\ \lim\limits_{y\to\infty} h'(y) = \frac{\Phi}{D_e} & \text{[asymptotic behaviour]}. \end{cases} \tag{7}$$

To better understand the role of each parameter, in the following we will consider three cases for the three **constant values** λ_ℓ, λ_m and λ_g defined in (3):

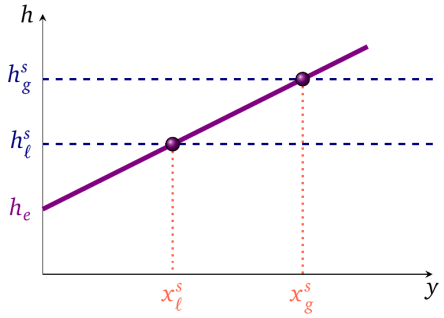


FIGURE 3. Steady solution without diffusion

(1) $\lambda_\ell = \lambda_m = \lambda_g = 0$ (model without diffusion)
(2) $\lambda_\ell, \lambda_m, \lambda_g > 0$ (model with positive diffusion)
(3) $\lambda_m = 0$ and $\lambda_\ell, \lambda_g > 0$ (model with a degenerate diffusion)

In the last case, which corresponds to the phase transition model, two cases have to be considered. Indeed, it will be seen that the mixture region can disappear for a sufficiently large gas diffusion coefficient λ_g.

2.1. Steady solution without diffusion

Proposition 2.1 (Steady solution without diffusion)**.** *When $\lambda_\ell = \lambda_m = \lambda_g = 0$, the steady enthalpy solution of* (6)-(7) *is*

$$h(y) = h_e + \frac{\Phi}{D_e} y. \tag{8}$$

Proof. Without diffusion we have to solve $D_e h'(y) = \Phi$ with $h(0) = h_e < h_\ell^s$. The solution is thus $h(y) = h_e + y\Phi/D_e$. □

Let us denote x_ℓ^s (resp. x_g^s) the liquid/mixture transition (resp. mixture/gas) without diffusion (see Figure 3). The fluid is in pure liquid phase for $y \le x_\ell^s$, in pure vapour phase for $y \ge x_g^s$ and a mixture at saturation if $x_\ell^s < y < x_g^s$ where

$$x_\ell^s = \frac{h_\ell^s - h_e}{\frac{\Phi}{D_e}}, \qquad x_g^s = \frac{h_g^s - h_e}{\frac{\Phi}{D_e}}. \tag{9}$$

2.2. Steady solution with a strictly positive diffusion

Without degeneracy of diffusion, the unique solution can be explicited as follows.

Proposition 2.2 (Steady solution with a positive diffusion)**.** *If $\lambda(h)$, defined in* (3)*, is piecewise positive constant for all h, the steady enthalpy solution of* (6)-(7) *is*

$$h(y) = \begin{cases} h_\ell(y) \overset{def}{=} C_1 + \dfrac{\Phi}{D_e} y + C_2 \exp\left(\dfrac{y}{\lambda_\ell / D_e}\right) & \text{if } y \le y_\ell^s \\ h_m(y) \overset{def}{=} C_3 + \dfrac{\Phi}{D_e} y + C_4 \exp\left(\dfrac{y}{\lambda_m / D_e}\right) & \text{if } y_\ell^s \le y < y_g^s \\ h_g(y) \overset{def}{=} h_g^s + \dfrac{\Phi}{D_e}(y - y_g^s) & \text{if } y \ge y_g^s \end{cases} \tag{10}$$

where the constants C_1, C_2, C_3, C_4, depending on y_ℓ^s and y_g^s, are

$$C_1 = \frac{h_e L_\ell^s + \frac{\Phi}{D_e}(y_\ell^s - 1)}{L_\ell^s - 1}, \qquad C_2 = h_e - C_1, \qquad L_\ell^s = \exp\left(\frac{y_\ell^s}{\lambda_\ell / D_e}\right)$$
$$C_3 = \frac{\left(\frac{\Phi}{D_e} y_\ell^s - h_\ell^s\right) M_g^s - \left(\frac{\Phi}{D_e} y_g^s - h_g^s\right) M_\ell^s}{M_\ell^s - M_g^s}, \quad C_4 = \frac{\left(\frac{\Phi}{D_e} y_g^s - h_g^s\right) - \left(\frac{\Phi}{D_e} y_\ell^s - h_\ell^s\right)}{M_\ell^s - M_g^s}, \quad M_\kappa^s = \exp\left(\frac{y_\kappa^s}{\lambda_m / D_e}\right),$$

for $\kappa = \ell$ or g and y_ℓ^s and y_g^s satisfies the continuity flux

$$\begin{cases} \lambda_\ell h'_\ell(y_\ell^s) = \lambda_m h'_m(y_\ell^s), \\ \lambda_m h'_m(y_g^s) = \lambda_g h'_g(y_g^s). \end{cases} \tag{11}$$

Proof. The non-degenerate diffusion, *i.e.* $\lambda(h) > 0$, leads to smooth solutions in the sense that discontinuities can not occur since order one distributions generated by the diffusive term and discontinuous solutions can not be compensated by order zero distribution. Furthermore, for continuous solutions, the transport term is then a diffuse measure leading to a continuous diffusion flux. Then, the continuous solution induces a mixture region and satisfies the flux continuity (11).

Due to the growth of the steady state solution (Φ and D_e are strictly positive), three regions, liquid, mixture, gas, are juxtaposed from low to high y values. On each region, an explicit solution solves the equation (6),

$$h(y) = \begin{cases} h_\ell(y) \stackrel{\text{def}}{=} C_1 + \frac{\Phi}{D_e} y + C_2 \exp\left(\frac{y}{\lambda_\ell / D_e}\right) & \text{if } y \le y_\ell^s \\ h_m(y) \stackrel{\text{def}}{=} C_3 + \frac{\Phi}{D_e} y + C_4 \exp\left(\frac{y}{\lambda_m / D_e}\right) & \text{if } y_\ell^s \le y < y_g^s \\ h_g(y) \stackrel{\text{def}}{=} C_5 + \frac{\Phi}{D_e} y + C_6 \exp\left(\frac{y}{\lambda_g / D_e}\right) & \text{if } y \ge y_g^s \end{cases} \tag{12}$$

where the unknows C_1, C_2, C_3, C_4, C_5, C_6, y_ℓ^s, y_g^s have to satisfy

- the enthalpy continuity conditions at inlet and transition points:

$$h_\ell(0) = h_e, \qquad h_\ell(y_\ell^s) = h_\ell^s, \tag{13}$$
$$h_m(y_\ell^s) = h_\ell^s, \qquad h_m(y_g^s) = h_g^s, \tag{14}$$
$$h_g(y_g^s) = h_g^s, \tag{15}$$

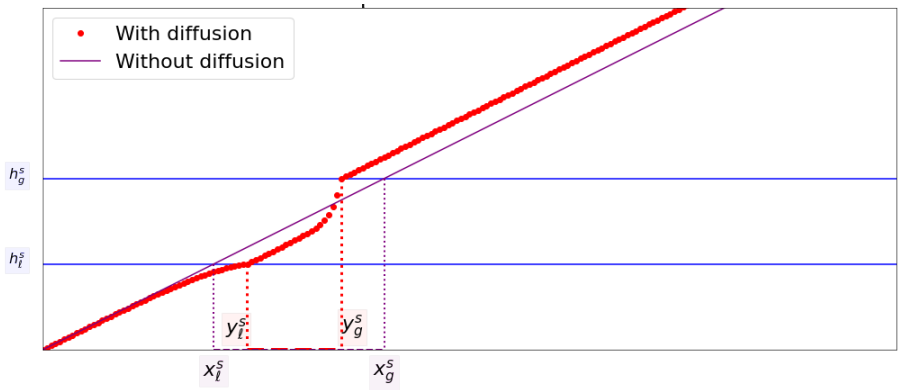
- the flux continuity conditions at transition points:

$$\lambda_\ell h'_\ell(y_\ell^s) = \lambda_m h'_m(y_\ell^s), \qquad \lambda_m h'_m(y_g^s) = \lambda_g h'_g(y_g^s), \tag{16}$$

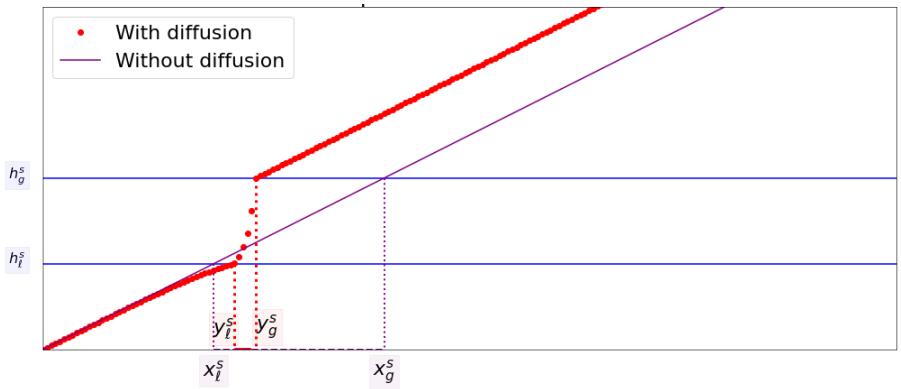
- the asymptotic behaviour

$$\lim_{y \to \infty} h'_g(y) = \frac{\Phi}{D_e}. \tag{17}$$

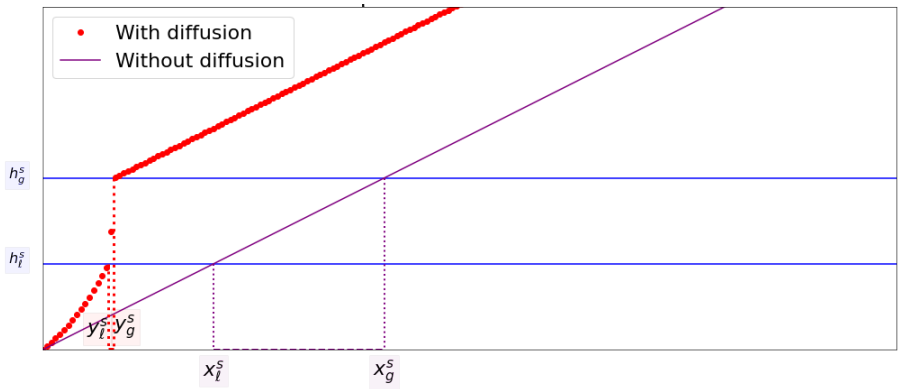
The conditions (13), (14), (15) and (17) allow to define C_1 to C_6 as functions of the transition points y_ℓ^s and y_g^s. These points are thus implicitly defined by (16) and finally we obtain the solution. We note that the transparent boundary condition (17) implies $C_6 = 0$ so that $y \mapsto h_g(y)$ is linear and compatible with the semi-infinite domain. Moreover, although $y \mapsto h'_g(y)$ does not involve λ_g, the flux continuity (16) does. □



(A) $\lambda_g = D_e$



(B) $\lambda_g = 2D_e$



(C) $\lambda_g = 4D_e$

FIGURE 4. Non degenerate diffusion for all h. Influence of λ_g.
Common parameters: $\Phi = D_e$, $2\lambda_\ell = D_e$, $10\lambda_m = D_e$

The Figure 4 shows the graph of $y \mapsto h(y)$ with $\lambda_m > 0$, that is a non-degenerate artificial diffusion in the mixture region, leading to a continuous solution. Depending on the increasing ratio λ_g/D_e when other parameters are fixed, the mixture region reduces and is pushed on the left. A stiffer solution appears at the gas transition. On the last plot (Figure 4c), the mixing region is very sharp and is not included in the large mixing region occurring without diffusion in the whole phases.

The first plot of the Figure 5 is the one of the Figure 4a. In the next figures is plotted a reduced diffusion coefficient λ_m. It is then exhibited the convergence of the enthalpy as λ_m goes to zero. A discontinuous solution is suggested at the limit. It occurs at the transition to the gas phase. For such diffusions in the liquid and gas phases, the mixture region is slightly affected by the decreasing diffusion λ_m and remains at the limit.

We will see in the following section that, when the diffusion is allowed to degenerate, *i.e.* the diffusive coefficient $\lambda(h)$ vanishes for values of h in the mixture region, solutions are not necessarily smooth and discontinuous weak solutions must be sought.

2.3. **Steady solution with degenerate diffusion**

The degenerate diffusion can lead to a jump in the diffusion flux at phase transition. For example, as the enthalpy flux is zero in the mixture region, a flux diffusion jump occurs at the interface with the mixture if the space derivative of the enthalpy is not zero at this transition in the liquid (or gas) phase. A Dirac measure follows and has to be compensated by an other Dirac measure in the transport term. A discontinuous solution is then necessary. Due to the growth of the steady state solution (Φ and D_e are positive) and to the presence of the liquid phase at inlet, a liquid phase is present for low y values, then follows a transition which can be liquid-mixture (followed by the transition mixture-gas) or liquid-gas. The two possible cases are presented below.

2.3.1. *Case I: Liquid/mixture/gas*

The solution with mixture is possible for a sufficiently low ratio $\frac{\lambda_g}{D_e}\frac{\Phi}{D_e}$ and is explicited in the following proposition.

Proposition 2.3 (Steady solution with degenerate diffusion: liquid/mixture/gas case)**.** *For all $\lambda_\ell > 0$, if*

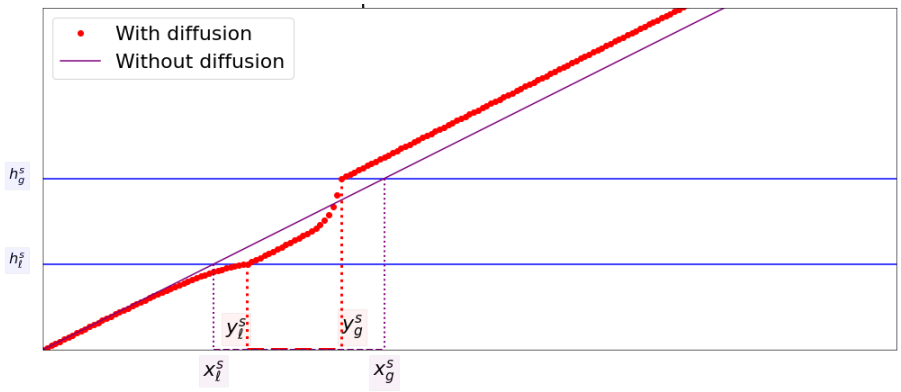
$$\frac{\lambda_g}{D_e}\frac{\Phi}{D_e} < h_g^s - h_\ell^s \tag{18}$$

the mixture zone is present and the solution of (6)-(7) *satisfies*

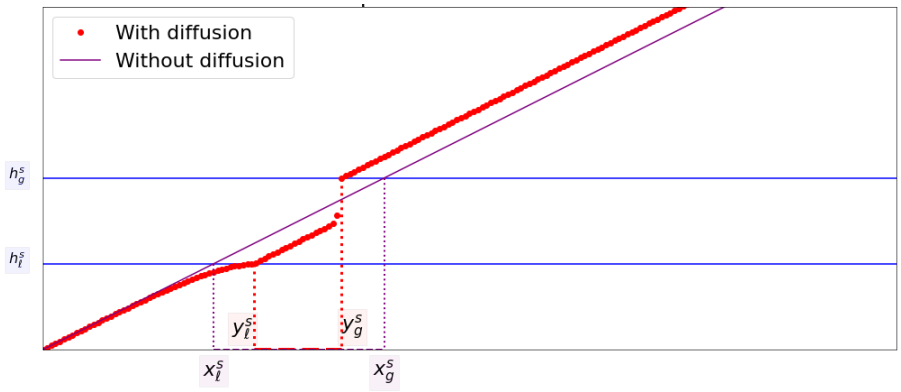
$$h(y) = \begin{cases} h_\ell(y) \overset{def}{=} h_e + \dfrac{\Phi}{D_e} y + \dfrac{\Phi}{D_e}\dfrac{\lambda_\ell}{D_e}\left[1 - \exp\left(\dfrac{y}{\lambda_\ell/D_e}\right)\right]\exp\left(\dfrac{-y_\ell^s}{\lambda_\ell/D_e}\right) & \text{if } y \le y_\ell^s \\ h_m(y) \overset{def}{=} h_\ell^s + \dfrac{\Phi}{D_e}(y - y_\ell^s) & \text{if } y_\ell^s \le y < y_g^s \\ h_g(y) \overset{def}{=} h_g^s + \dfrac{\Phi}{D_e}(y - y_g^s) & \text{if } y \ge y_g^s \end{cases} \tag{19}$$

where

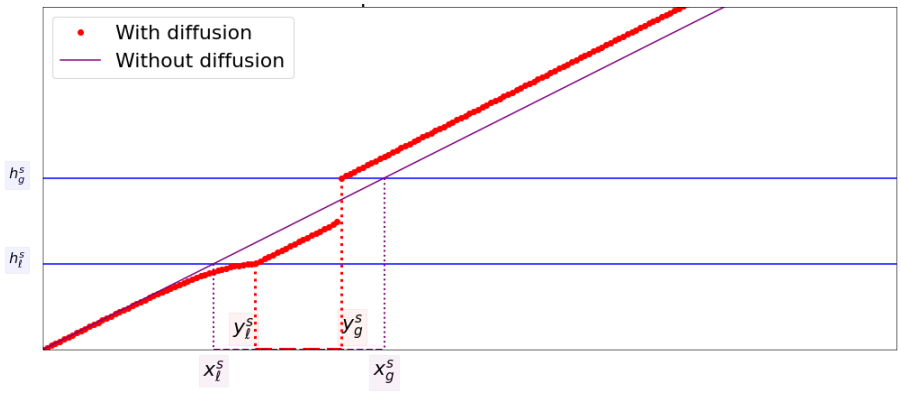
- *the position y_ℓ^s separating the liquid phase to the mixture phase is implicitly defined by $h_\ell(y_\ell^s) = h_\ell^s$;*
- *the position y_g^s separating the mixture phase and the gas phase is computed w.r.t. y_ℓ^s by $y_g^s = y_\ell^s + \frac{D_e}{\Phi}(h_g^s - h_\ell^s) - \frac{\lambda_g}{D_e}$.*



(A) $\frac{\lambda_m}{D_e} = 0.1$



(B) $\frac{\lambda_m}{D_e} = 0.025$



(C) $\frac{\lambda_m}{D_e} = 0.0125$

FIGURE 5. Non degenerate diffusion for all h. Convergence of $\lambda_m \to 0$.
Common parameters: $\Phi = D_e$, $2\lambda_\ell = D_e$, $\lambda_g = D_e$

Proof. As for the nondegenrate case, on each phase or mixture, the integration of the solution gives

$$h(y) = \begin{cases} h_\ell(y) \stackrel{\text{def}}{=} C_1 + \dfrac{\Phi}{D_e} y + C_2 \exp\left(\dfrac{y}{\lambda_\ell / D_e}\right) & \text{if } y \le y_\ell^s, \\ h_m(y) \stackrel{\text{def}}{=} C_3 + \dfrac{\Phi}{D_e}(y - y_\ell^s) & \text{if } y_\ell^s \le y < y_g^s, \\ h_g(y) \stackrel{\text{def}}{=} C_4 + \dfrac{\Phi}{D_e}(y - y_g^s) + C_5 \exp\left(\dfrac{y}{\lambda_g / D_e}\right) & \text{if } y \ge y_g^s. \end{cases} \tag{20}$$

The five constants $C_1 \cdots C_5$ have to be determine and a particular choice will be made to define a unique viscosity solution. Searching a solution of (6) with mixture, the transition liquid-mixture induces a free dissipative flux in the mixture. A positive jump on dissipative flux can not be compensated by the enthalpy jump on the transport term since the enthalpy is increasing. As a matter of fact, if we denote $[\![h]\!]$ the enthalpy jump at the liquid-mixture interface, the jump relation of (6) leads to

$$D_e [\![h]\!] - 0 \times h'(y_\ell^{s,+}) + \lambda_\ell h'(y_\ell^{s,-}) = 0,$$

that is

$$\lambda_\ell h'(y_\ell^{s,-}) = -D_e [\![h]\!].$$

Since $y \mapsto h(y)$ has to be increasing, we should have $[\![h]\!] \le 0$. Hence, in the present case, the only possibility is to have both $[\![h]\!] = 0$ and $h'(y_\ell^{s,-}) = 0$. The inlet Dirichlet boundary condition and the Dirichlet condition $h(y_\ell^s) = h_\ell^s$ define the constants C_1 and C_2 parametrized with respect to y_ℓ^s. Then, the additional Neumann boundary condition at y_ℓ^s defines implicitly the unique value of y_ℓ^s, C_1 and C_2, that is the unique solution in the known liquid phase.

The expressions of the solution in the mixture region follow by the continuity of the enthalpy at the interface with liquid, thus $C_3 = h_\ell^s$. Furthermore, since $C_5 = 0$ by virtue of the asymptotic behavior (7),

$$h_g(y) = C_4 + \frac{\Phi}{D_e}(y - y_g^s).$$

The difficulty is then to define the transition point y_g^s and the constant C_4. The jump relation at this point is obtained in the same way as for the point y_ℓ^s, but this time, the two jumps can compensate each other:

$$D_e [\![h]\!] - \lambda_g h'(y_g^{s,+}) + 0 \times h'(y_g^{s,-}) = 0,$$

that is

$$[\![h]\!] = \frac{\lambda_g}{D_e} \frac{\Phi}{D_e},$$

which has to be lower than the maximal gap $h_g^s - h_\ell^s$.

There is an infinity of solutions to such a problem in the sense of distribution. The transition y_g^s separating enthalpy lower than h_g^s to upper can cover an interval, that is,

$$h_g(y_g^s) \in \left[h_g^s, h_g^s + \frac{\lambda_g}{D_e} \frac{\Phi}{D_e} \right[.$$

Thus,

$$h_m(y_g^s) = h_g(y_g^s) - [\![h]\!] = h_g(y_g^s) - \frac{\lambda_g}{D_e} \frac{\Phi}{D_e}.$$

In order to define an unique solution, we choose a solution corresponding to the smallest mixture region, that is with the smallest value of enthalpy $h_g(y_g^s) = C_4 = h_g^s$ in the gas region at the transition. The reader

can convince himself (see Figure 5) that this choice corresponds to the viscosity solution by studying the limit of the viscous solution obtained in Proposition 2.2. This implies that

$$h_m(y_g^s) = h_g^s - \frac{\lambda_g}{D_e}\frac{\Phi}{D_e}$$

and the position y_g^s follows

$$y_g^s = y_\ell^s + \frac{D_e}{\Phi}(h_g^s - h_\ell^s) - \frac{\lambda_g}{D_e}.$$

□

Remark 2.4. Gas diffusion reduces the mixture region for steady solution since

$$(y_g^s - y_\ell^s) = (x_g^s - x_\ell^s) - \frac{\lambda_g}{D_e}$$

where x_κ^s are the transition points without diffusion defined in (9). We remark also that $y_\ell^s > x_\ell^s$ if $\lambda_\ell > 0$. We point out that if (and only if) the condition

$$\frac{\lambda_g}{D_e}\frac{\Phi}{D_e} \leq h_g^s - h_\ell^s$$

is not fulfilled, a transition liquid-gas has to be written (*cf.* the following section).

The Figure 6 shows some steady solutions when (18) is fulfilled so that the mixture zone is present. The mixture region is always smaller than the one obtained without diffusion, *i.e.* $y_g^s - y_\ell^s \leq x_g^s - x_\ell^s$. More precisely, we observe:

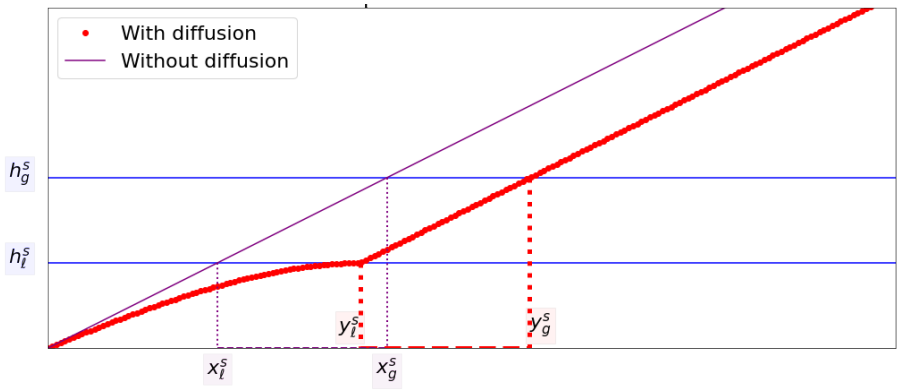
- The Figure 6a shows the steady solution when $\Phi = D_e$, $\lambda_\ell = 2D_e$ and $\lambda_g = 0$. We can see the effect of the diffusion only in the liquid phase: the liquid/mixture transition occurs at $y_\ell^s > x_\ell^s$ and the slope of h_ℓ at y_ℓ^s is zero.
- Figure 6b shows the steady solution when $\Phi = D_e$, $\lambda_\ell = 0$ and $\lambda_g = D_e$. We can see the effect of the diffusion only in the vapour phase: the mixture/vapour transition occurs at $y_g^s < x_g^s$; the width of the mixture zone is reduced w.r.t. the case without diffusion and there is a jump on the enthalpy h at transition y_g^s.
- Finally Figure 6c shows the steady solution when $\Phi = D_e$, $2\lambda_\ell = D_e$ and $\lambda_g = D_e$. We can see the effect of the diffusion both in the liquid and in the vapour phases: the liquid/mixture transition occurs at $y_\ell^s > x_\ell^s$ and the slope of h_ℓ at y_ℓ^s is zero; the mixture/vapour transition occurs at $y_g^s < x_g^s$; the width of the mixture zone is reduced w.r.t. the case without diffusion as observed in the remark 2.4 and there is a jump on the enthalpy h at transition y_g^s.

2.3.2. *Case II: Liquid/gas (no mixture, Stefan problem)*

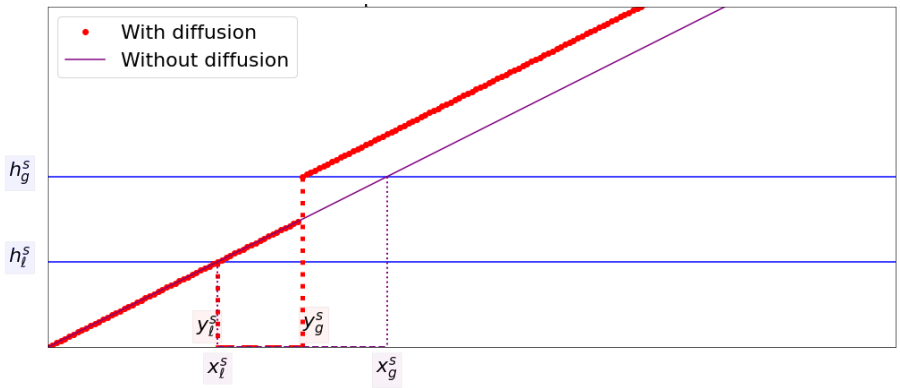
The solution without mixture occurs for a sufficiently large ratio $\frac{\lambda_g}{D_e}\frac{\Phi}{D_e}$ and is explicited in the following proposition.

Proposition 2.5 (Steady solution with degenerate diffusion: liquid/gas case)**.** *For all $\lambda_\ell > 0$, if*

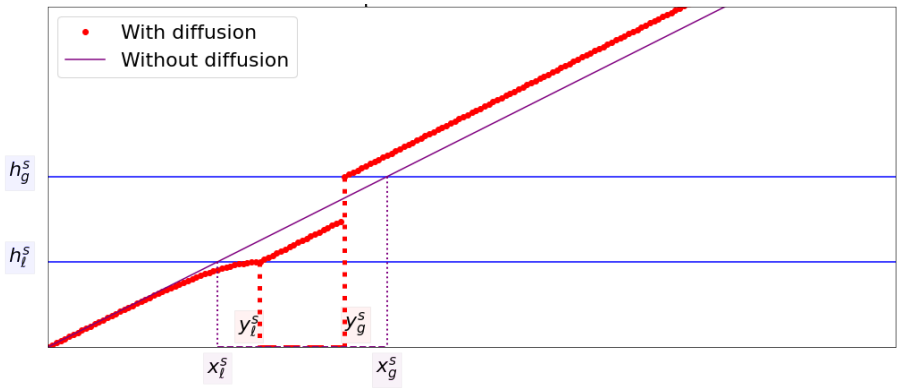
$$\frac{\lambda_g}{D_e}\frac{\Phi}{D_e} \geq h_g^s - h_\ell^s \tag{21}$$



(A) Diffusion only in the liquid phase: $\lambda_\ell = 2D_e$ and $\lambda_g = 0$. We observe the influence on y_ℓ^s and the slope at y_ℓ^s.



(B) Diffusion only in the gas phase: $\lambda_g = D_e$ and $\lambda_\ell = 0$. We observe the influence on Δy_κ^s the width of the mixture zone and jump at y_g^s



(C) Diffusion both in liquid and vapour phases: $2\lambda_\ell = D_e$ and $\lambda_g = D_e$. We observe the existence of the mixture zone at steady state

FIGURE 6. Degenerate diffusion. Case I: Mixture zone at steady state. Common parameters: $\Phi = D_e$ and $\lambda_m = 0$.

the mixture zone is not present and the solution of (6)-(7) *satisfies*

$$h(y) = \begin{cases} h_\ell(y) \overset{def}{=} h_e + \dfrac{\Phi}{D_e} y + \left[(h_g^s - h_\ell^s) - \left(\dfrac{\lambda_g}{D_e} - \dfrac{\lambda_\ell}{D_e} \right) \dfrac{\Phi}{D_e} \right] \left[1 - \exp\left(\dfrac{y}{\lambda_\ell / D_e} \right) \right] \exp\left(\dfrac{-y^s}{\lambda_\ell / D_e} \right) & \text{if } y < y^s \\ h_g(y) = h_g^s + \dfrac{\Phi}{D_e}(y - y^s) & \text{if } y > y^s \end{cases}$$

where the position y^s *separating the liquid phase to the gas phase is implicitly defined by* $h_\ell(y^s) = h_\ell^s$.

Proof. As shown in the proof of Proposition 2.3, there isn't viscosity solutions to (6)-(7) with mixture if

$$\frac{\lambda_g}{D_e} \frac{\Phi}{D_e} \geq h_g^s - h_\ell^s.$$

On each phase, the integration of the differential equation gives

$$h(y) = \begin{cases} h_\ell(y) \overset{\text{def}}{=} C_1 + \dfrac{\Phi}{D_e} y + C_2 \left[1 - \exp\left(\dfrac{y}{\lambda_\ell / D_e} \right) \right] & \text{if } y < y^s \\ h_g(y) \overset{\text{def}}{=} C_3 + \dfrac{\Phi}{D_e}(y - y^s) + C_4 \exp\left(\dfrac{y}{\lambda_g / D_e} \right) & \text{if } y > y^s \end{cases} \tag{22}$$

The four constants $C_1 \dots C_4$ and the transition point y^s have to be determined. The inlet Dirichlet boundary condition and the asymptotic condition (7) define the constants $C_1 = h_e$ and $C_4 = 0$.

A particular choice will be made to define an unique viscosity solution. Such a viscosity solution has to satisfy

$$\begin{cases} h(y^{s,-}) = h_\ell^s, \\ h(y^{s,+}) = h_g^s. \end{cases}$$

This implies that $C_3 = h_g^s$ and

$$C_2 = \frac{h_\ell^s - h_e - \frac{\Phi}{D_e} y^s}{1 - \exp\left(\frac{y^s}{\lambda_\ell / D_e} \right)}.$$

The jump relation at the transition point y^s reads

$$D_e(h_g^s - h_\ell^s) = \lambda_g h'(y^{s,+}) - \lambda_\ell h'(y^{s,-}).$$

Since $C_4 = 0$, $h_g'(y) = \frac{\Phi}{D_e}$ for all y so that

$$\frac{\lambda_\ell}{D_e} h_\ell'(y^{s,-}) = \frac{\lambda_g}{D_e} \frac{\Phi}{D_e} - (h_g^s - h_\ell^s),$$

which is positive under the assumption (21) and is then compatible with the increasing enthalpy h.

Computing the derivative of the enthalpy in the liquid phase, we obtain an implicit definition of the position y^s:

$$\left[(h_g^s - h_\ell^s) + \frac{\Phi}{D_e} \left(\frac{\lambda_\ell}{D_e} - \frac{\lambda_g}{D_e} \right) \right] \left(\exp\left(\frac{y^s}{\lambda_\ell / D_e} \right) - 1 \right) = \left[(h_e - h_\ell^s) + \frac{\Phi}{D_e} y^s \right] \exp\left(\frac{y^s}{\lambda_\ell / D_e} \right).$$

We now prove that this value is unique. Let us we denote $t \overset{\text{def}}{=} \exp\left(\frac{y^s}{\lambda_\ell / D_e} \right) \geq 1$, $C \overset{\text{def}}{=} (h_g^s - h_\ell^s) + \frac{\Phi}{D_e} \left(\frac{\lambda_\ell}{D_e} - \frac{\lambda_g}{D_e} \right)$ and $R = \frac{\Phi}{D_e} \frac{\lambda_\ell}{D_e}$. The equation on the flux continuity can be rewritten as $C(t-1) = [(h_e - h_\ell^s) + R \ln(t)]\, t$, thus

the position y^s is implicitly defined by the zero of the function

$$f(t) = (C + h_\ell^s - h_e)t - Rt\ln(t) - C, \qquad t \geq 1.$$

Since $f(1) = h_\ell^s - h_e > 0$, $f(t) \xrightarrow[t\to\infty]{} -\infty$, and $f''(t) = -R/t < 0$, the function f admits an unique zero for $t > 1$. □

Figure 7 shows some steady solutions when $\frac{\lambda_g}{D_e}\frac{\Phi}{D_e} \geq h_g^s - h_\ell^s$ so that the mixture zone is not present.

- Figure 7a shows the steady solution when $\Phi = D_e$, $\lambda_\ell = D_e$ and $\lambda_g = 2D_e$. This corresponds to the smallest choice of λ_g such that the jump is equals to $h_g^s - h_\ell^s$. The mixture is not present, the liquid/vapour transition occurs at y^s and the slope of h_ℓ at y_ℓ^s is zero.
- Figure 7b shows the steady solution when $\Phi = D_e$, $\lambda_\ell = D_e$ and $\lambda_g = 4D_e$. The jump is equals to $h_g^s - h_\ell^s$, the mixture is not present, the liquid/vapour transition occurs at y^s and the slope of h_ℓ at y_ℓ^s is strictly positive. When $\lambda_\ell \to 0$ the slope tends to $+\infty$:

$$h'_\ell(y^{s,-}) = \frac{\frac{\lambda_g}{D_e}\frac{\Phi}{D_e} - (h_g^s - h_\ell^s)}{\frac{\lambda_\ell}{D_e}} \qquad \left[\geq 0 \text{ and } \xrightarrow[\lambda_\ell\to 0]{} +\infty\right]$$

- Finally Figure 7c shows the steady solution when $2\Phi = D_e$, $\lambda_\ell = 0$ and $\lambda_g = 5D_e$. The mixture is not present, the liquid/vapour transition occurs at y^s. Since $\lambda_\ell = 0$ there is a jump greater to $h_g^s - h_\ell^s$ so that we have a jump in liquid phase that corresponds to a jump on temperature and $h_\ell(y^s) < h_\ell^s$:

$$[\![h]\!](y^s) = h_g^s - h_\ell(y^s) = \frac{\lambda_g}{D_e}\frac{\Phi}{D_e} \geq h_g^s - h_\ell^s.$$

2.4. **Generalization to negative power densities**

An enthalpy jump was exhibited at the gas transition point y_g^s. The gas phase does not play a different role from the liquid one: it is the situation of a heating a fluid (that is $\Phi > 0$) that induces an increasing solution in the direction of flow with a jump localized at the last transition point.

In the case of a cooling heat exchanger (that is $\Phi < 0$), the fluid is injected in gas phase ($h_e > h_g^s$): the stationary solution is thus decreasing in the direction of flow and it presents:

- a jump at the mixture-liquid transition point in case of

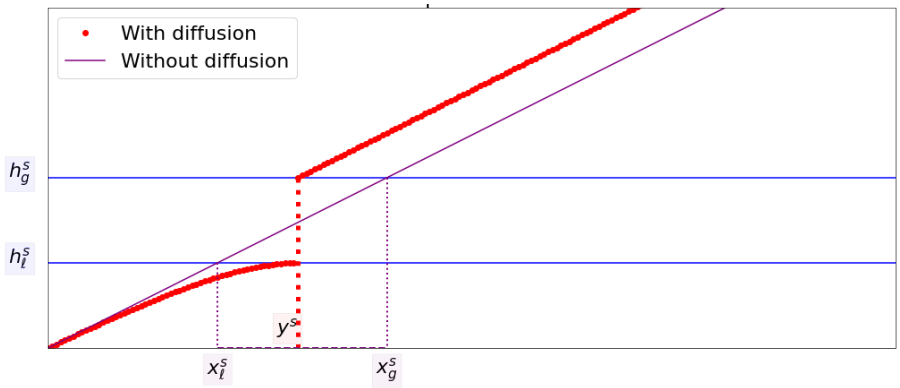
$$\frac{\lambda_g}{D_e}\frac{|\Phi|}{D_e} < h_g^s - h_\ell^s;$$

- a discontinuity at the gas-liquid transition point otherwise.

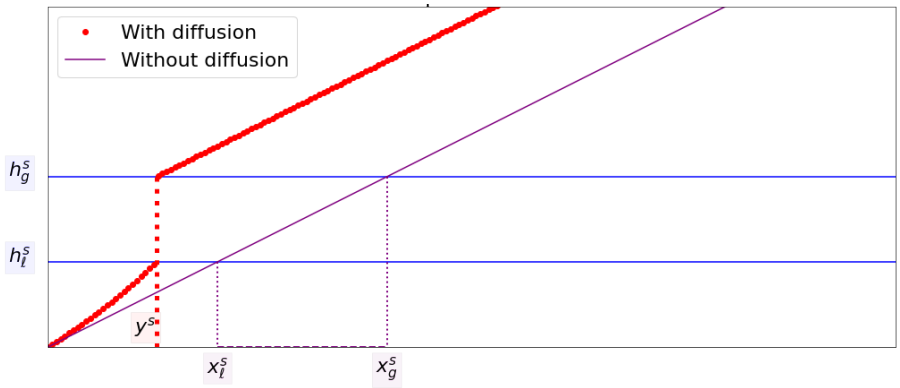
2.5. **Link to the Stefan problems**

The solution exhibited in the Proposition 2.5 is exactly the stationary solution of the following stationary Stefan problem, a free boundary problem. That is, find the position y^s so that the over-determinate elliptic problem is satisfied:

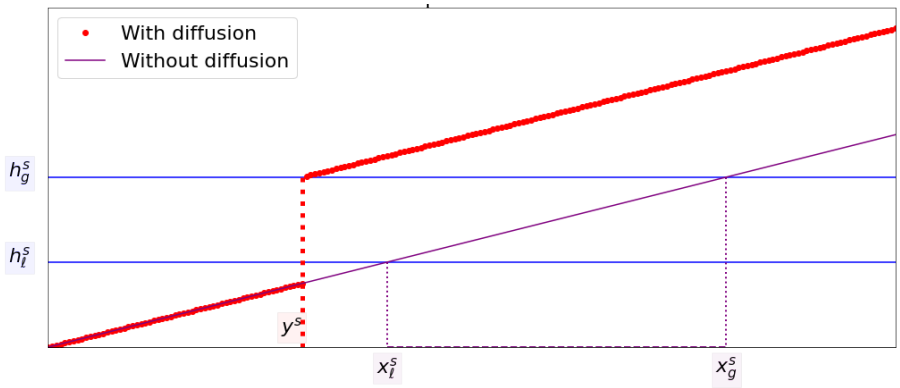
$$\begin{cases} D_e h' - \lambda_\ell h'' = \Phi \text{ in }]0, y^s[\\ h(0) = h_e < h_\ell^s, \quad h(y^{s,-}) = h_\ell^s, \\ \\ D_e h' - \lambda_g h'' = \Phi \text{ in }]y^s, +\infty[\\ h(y^{s,+}) = h_\ell^s, \quad \lim\limits_{y\to\infty} h'(y) = \frac{\Phi}{D_e}, \\ \\ \lambda_g h'(y^{s,+}) - \lambda_\ell h'(y^{s,-}) = D_e(h_g^s - h_\ell^s). \end{cases} \tag{23}$$



(A) $\Phi = D_e$, $\lambda_\ell = D_e$, $\lambda_g = 2D_e$.
Disappearing of the mixture zone at steady state



(B) $\Phi = D_e$, $\lambda_\ell = D_e$, $\lambda_g = 4D_e$.
No mixture zone, the slope at $y^{s,-}$ can be $> \frac{\Phi}{D_e}$



(C) $2\Phi = D_e$, $\lambda_\ell = 0$, $\lambda_g = 5D_e$.
No diffusion in liquid phase, jump in liquid phase at y^s

FIGURE 7. Degenerate diffusion. Case II: No mixture zone at steady state.
Common parameter: $\lambda_m = 0$.

Written with the temperature as unknown, we obtain the classic stationary Stefan problem, with a continuous temperature at the transition and a jump condition on thermal flux, as it can be found in [12],

$$\begin{cases} c_{p,\ell} D_e T' - \omega_\ell T'' = \Phi \text{ in }]0, y^s[\\ T(0) = T_e < T^s, \quad T(y^{s,-}) = T^s, \\ \\ c_{p,g} D_e T' - \omega_g T'' = \Phi \text{ in }]y^s, +\infty[\\ T(y^{s,+}) = T^s, \quad \lim\limits_{y\to\infty} T'(y) = \frac{\Phi}{c_{p,g} D_e}, \\ \\ \omega_g T'(y^{s,+}) - \omega_\ell T'(y^{s,-}) = D_e(h_g^s - h_\ell^s). \end{cases} \tag{24}$$

Note that the last relation of (24) describes the jump on heat flux due to latent heat.

This problem can be solved even if (18) holds, but the solution would loose a physical meaning due to inconsistent temperature with respect to the considered phase. As a matter of fact, the last relation of (24) gives

$$\omega_\ell T'(y^{s,-}) = \lambda_g \frac{\Phi}{D_e} - D_e(h_g^s - h_\ell^s) < 0 \text{ under (18)},$$

which induces a temperature above T^s in a neighborhood of y^s in the liquid phase.

To summarize, the Stefan formulations (23) or (24) have to be used only under (21) contrary to (6)-(7) which is valid under (21) and (18). Under (21), the solutions of the formulations are the same; under (18), the Stefan solution of (23) or (24) isn't physical. That is why the PDE formulation (6)-(7) has to be preferred for numerical solving as it is shown in Appendix A, extended to the non-stationary case.

3. Conclusion and perspectives

It has been shown the expression of the physical stationary solution for a one dimensional degenerate nonlinear elliptic problem modelling enthalpy subjected to a flow and thermal diffusion with phase change. The expression of the solution depends strongly on the ratio of heat deposit with flow rate to compare with the ratio of flow rate and diffusion coefficient in the gas phase (see (21)). It is notable that discontinuous solutions appear in the context of a modelization with diffuse interface. Furthermore, this mixture region which always exists without thermal diffusion, can disappear depending on the ratio (21). The selected solution is the viscosity solution leading to a minimal mixture region. The higher the gas diffusion coefficient, the smaller the mixing region, until it disappears.

Some unsteady simulations are proposed in the appendix but a more thorough numerical study is to follow for the full model proposed (2) in a future work.

A. Numerical Time-Dependent solution of the constant-density model

The time-dependent 1D model (4) can be written in a non-conservative formulation as

$$\begin{cases} \partial_t \rho + \partial_y(\rho v) = 0, \\ \rho \partial_t h + \rho v \partial_y(h) = \Phi + \partial_y(\lambda(h)\partial_y h), \end{cases} \in \mathbb{R}^+ \times \mathbb{R}^+.$$

In the stationary case, the model does not depend on the expression of the density with respect to the enthalpy. It is remarkable that whatever the density expression, the time-asymptotic solution depends only on the uniform flow rate. In the physical model, the density is assumed to be a continuous function of the enthalpy h, piecewise defined for each phase or mixture. In the transient regime, the law describing the density with respect to the enthalpy affects the unsteady solution. With the EoS describing a mixture at saturation as well as the pure phases, the density is a continuous and piecewise defined function of the enthalpy (and the constant

thermodynamic pressure p_*). For simplicity, we choose here to exhibit the transient regime in the case of an artificially uniform in space and constant in time density. Let us consider $\rho(h) = r > 0$ constant for all h. The system becomes

$$\begin{cases} \partial_y v = 0, \\ r\partial_t h + rv\partial_y h = \Phi + \partial_y(\lambda(h)\partial_y h) \end{cases} \quad \text{in } [0; +\infty),$$

with the boundary condition $(rv)(0) = D_e > 0$ and $h(0) = h_e < h_\ell^s$.

Thus $v(y) = v(0)$ for all $y \in \mathbb{R}^+$ and $(\rho v)(y) = rv(0) = D_e$ for all $y \in \mathbb{R}^+$. The enthalpy equation can be written as

$$r\partial_t h + D_e\partial_y h - \partial_y(\lambda(h)\partial_y h) = \Phi.$$

The unsteady model, written on the domain $(t, x) \in]0, T[\times]0, +\infty[$, reads

$$r\partial_t h + D_e\partial_y h - \partial_{yy}(L(h)) = \Phi, \qquad L(h) \stackrel{\text{def}}{=} \begin{cases} \lambda_\ell(h - h_\ell^s) & \text{if } h \le h_\ell^s \\ 0 & \text{if } h_\ell^s < h < h_g^s \\ \lambda_g(h - h_g^s) & \text{if } h \ge h_g^s \end{cases}$$

with $L'(h) = \lambda(h)$, associated to the following boundary and initial conditions,

$$\begin{cases} h(t, 0) = h_e < h_\ell^s & \text{[inlet Dirichlet boundary condition]} \\ \lim\limits_{y\to\infty} \partial_y h(t, \cdot) = \frac{\Phi}{D_e} & \text{[asymptotic behaviour]} \\ h(0, \cdot) = h_0(\cdot) < h_\ell^s & \text{[initial condition].} \end{cases}$$

We define δt the time step and $h^n \approx h(n\delta t)$. It is then essential to numerically solve this evolution problem by a fully implicit scheme,

$$r\frac{h^{n+1} - h^n}{\delta t} + D_e\partial_y h^{n+1} - \partial_{yy}(L(h^{n+1})) = \Phi \quad \text{in } \mathbb{R}^+,$$

associated to a gradient scheme [6], in order to produce a viscosity solution. All the simulations respects the enthalpy properties at transition (jump relations), namely a jump of enthalpy (when it has to be) at the transition to the gas phase with an enthalpy close to h_g^s (up to the order of space discretization step) at the first node where enthalpy is upper than h_g^s. The increasing solution is also observed on the discrete solution avoiding spurious numerical oscillations.

Note that the solution converges, when time goes to infinity, to the analytic solution explicited in the Propositions 2.2 and 2.3. Note also that mixture appears in transient regime even if the asymptotic solution solves the Stefan problem with a transition liquid/gas, as on the Figure 9.

On Figures 8 and 9, the initial condition is in liquid phase, then appears the mixture and finally the gas phase. The parameters of Figure 8 satisfy condition (18) with mixture on time-asymptotic solution.

- Figure 8a shows the solution without diffusion in the gas phase and the solution shows the characteristic transition liquid-mixture with a zero space derivative in the liquid at the motionless transition.
- Figure 8b shows the solution without diffusion in the liquid phase and the solution shows the characteristic transition mixture-gas with a growing jump of enthalpy at the moving transition.
- Finally Figure 8c shows the solution with diffusion in the liquid and gas phase gathering the two previous behaviors.

The parameters of Figure 9 satisfy condition (21) without mixture on the time-asymptotic solution.

- Figure 9a shows the apparition of mixture then gas phase. The jump on the enthalpy is increasing along time and is asymptotically the maximal jump $h_g^s - h_\ell^s$. This is the critical case where mixture disappears asymptotically with a zero space derivative in the liquid at the transition.
- Figure 9b shows the same phenomena with a bigger diffusion in the gas phase leading to positive slope in the liquid at the transition for sufficient large times.

- Finally Figure 9c shows the particular case of a zero diffusion in the liquid phase leading to an increasing jump at the transition point to the gas phase. For sufficiently large time, the jump exceeds the value $h_g^s - h_\ell^s$ due to the lack of diffusion in the liquid phase which behaves like the mixture.

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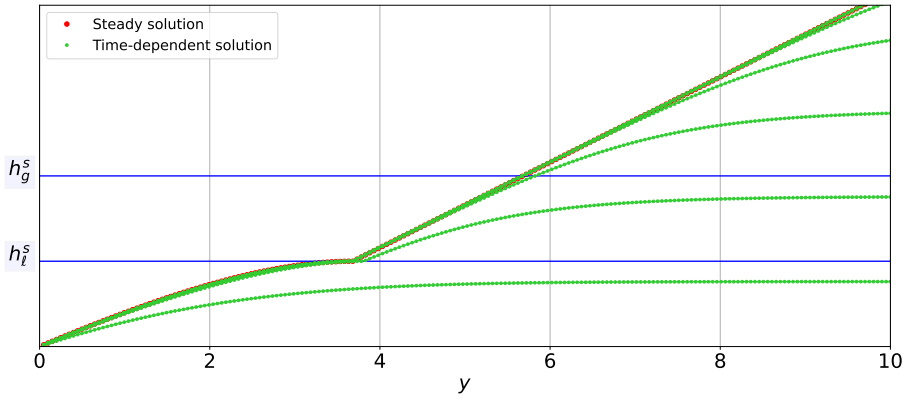
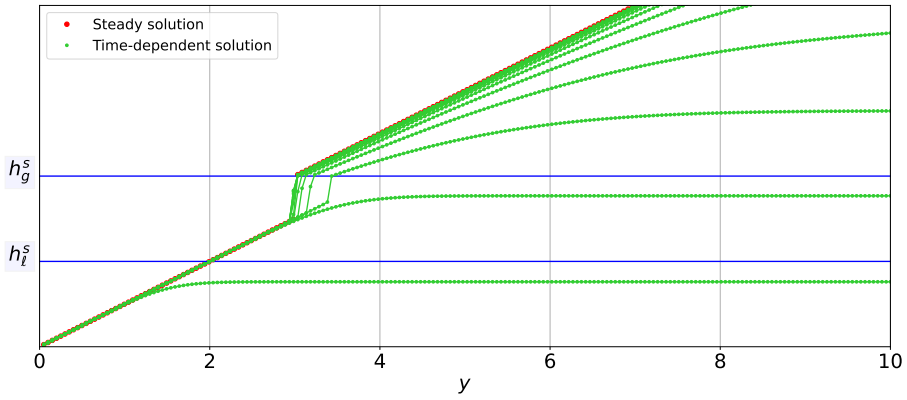
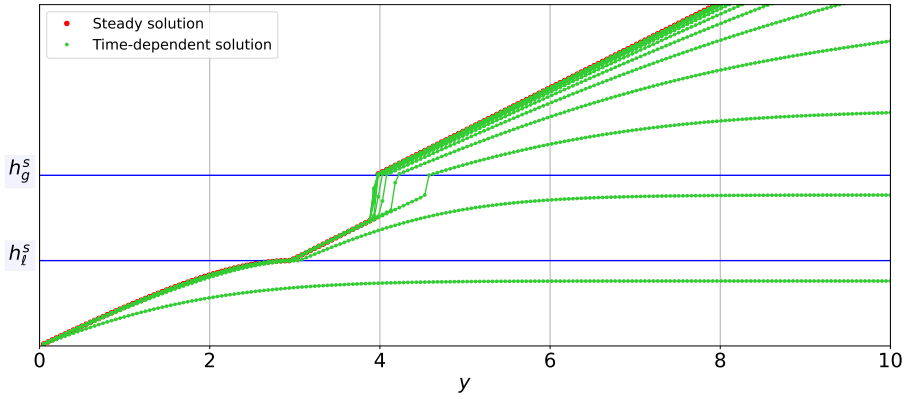


(A) Diffusion only in the liquid phase: $\lambda_\ell = 2D_e$, $\lambda_g = 0$.
We observe the influence on y_ℓ^s and the slope at y_ℓ^s.

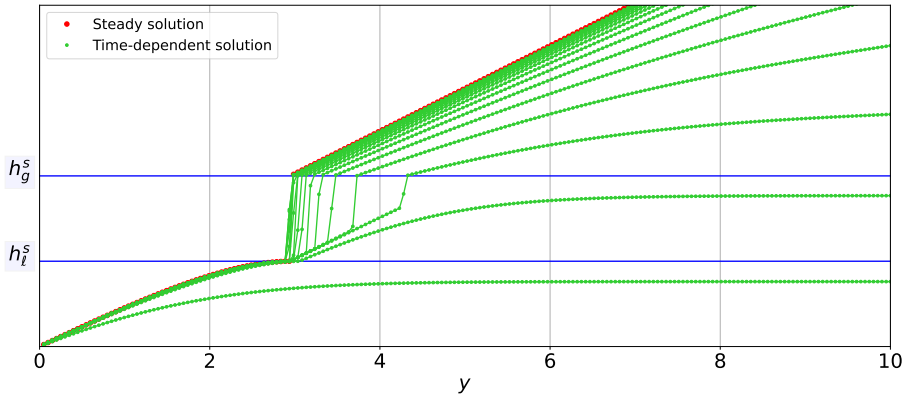


(B) Diffusion only in the gas phase: $\lambda_\ell = 0$, $\lambda_g = D_e$.
We observe the influence on Δy_κ^s the width of the mixture zone and jump at y_g^s.

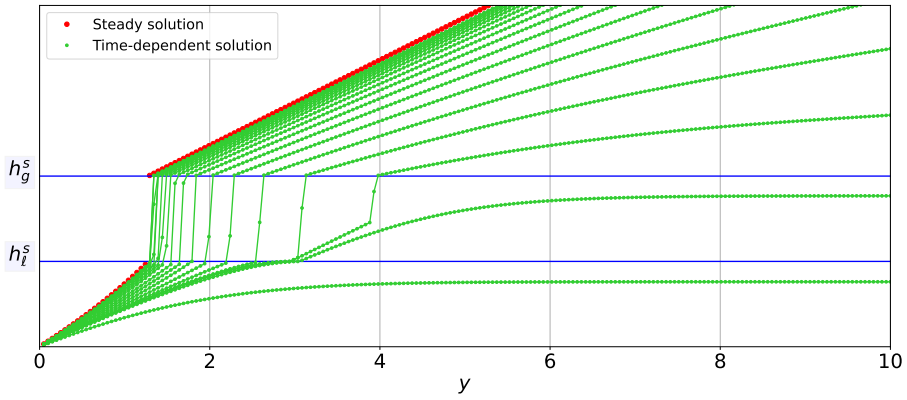


(C) Diffusion both in liquid and vapour phases: $\lambda_\ell = D_e$, $\lambda_g = D_e$.
We observe the existence of the mixture zone at steady state.

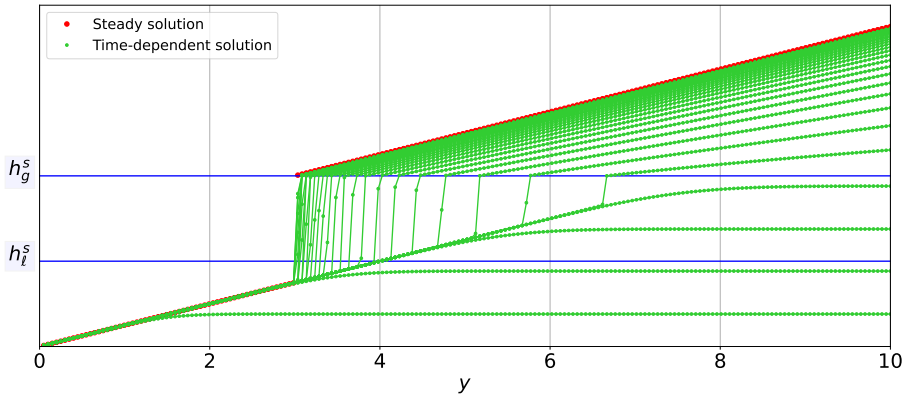
FIGURE 8. Mixture zone at steady state. Common parameters: $\Phi = D_e$, $\lambda_m = 0$.



(A) $\Phi = D_e$, $\lambda_\ell = D_e$, $\lambda_g = 2D_e$.
We observe the disappearing of the mixture zone at steady state



(B) $\Phi = D_e$, $\lambda_\ell = D_e$, $\lambda_g = 4D_e$.
We observe the disappearing of the mixture zone and the slope at $y^{s,-}$ can be $> \frac{\Phi}{D_e}$



(C) $2\Phi = D_e$, $\lambda_\ell = 0$, $\lambda_g = 5D_e$.
Diffusion only in vapour phase, no mixture zone, jump in liquid phase at y^s

FIGURE 9. Diffusion both in liquid and vapour phases. No mixture zone at steady state.