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# MEAN FIELD THEORY FOR A GENERAL CLASS OF SHORT-RANGE INTERACTION FUNCTIONALS

GUY BOUCHITTÉ AND RAJESH MAHADEVAN

ABSTRACT. In models of  $N$  interacting particles in  $\mathbb{R}^d$  as in Density Functional Theory or crowd motion, the repulsive cost is usually described by a two-point function  $c_\varepsilon(x, y) = \ell\left(\frac{|x-y|}{\varepsilon}\right)$  where  $\ell : \mathbb{R}_+ \rightarrow [0, \infty]$  is decreasing to zero at infinity and parameter  $\varepsilon > 0$  scales the interaction distance. In this paper we identify the mean-field energy of such a model in the short-range regime  $\varepsilon \ll 1$  under the sole assumption that  $\exists r_0 > 0 : \int_{r_0}^\infty \ell(r)r^{d-1} dr < +\infty$ . This extends recent results [12, 13, 17] obtained in the homogeneous case  $\ell(r) = r^{-s}$  where  $s > d$ .

**Keywords:** empirical measures, non-local functionals,  $\Gamma$ -convergence, mean-field energy, sub-additivity

**Mathematics Subject Classification:** 49J45, 49K21, 49N15, 60B10, 70-10, 82B21

## 1. INTRODUCTION

We consider a repulsive interaction function on  $(\mathbb{R}^d)^N$  of the kind

$$(1.1) \quad c_N^\varepsilon(x_1, \dots, x_N) = \sum_{i \neq j} \ell\left(\frac{|x_i - x_j|}{\varepsilon}\right).$$

where:

- $N$  is the number of particles in  $\mathbb{R}^d$ ;
- $\varepsilon > 0$  scales the interaction distance between particles.
- the two-particle cost  $\ell : [0, +\infty] \rightarrow [0 + \infty]$  satisfies:

(H1)  $\ell$  is l.s.c. and  $\ell(0) > 0$  ( $\ell(0) = +\infty$  is allowed)

(H2)  $\exists r_0 \geq 0$  such that  $\ell$  is finite and non increasing on  $[r_0, +\infty)$  and  $\lim_{r \rightarrow \infty} \ell(r) = 0$ .

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In the whole paper, we denote by  $\overline{\Omega}$  the closure of a smooth domain  $\Omega \subset \mathbb{R}^d$  where the  $N$  particles are located. We speak of a *confined system* when  $\overline{\Omega}$  is compact (container). Given a continuous exterior potential  $U : \overline{\Omega} \rightarrow \mathbb{R}$ , we consider the finite dimensional problem:

$$(1.2) \quad \mathcal{E}_N^\varepsilon(\Omega, U) := \inf \left\{ h_N c_N^\varepsilon(x_1, \dots, x_N) + \frac{1}{N} \sum_{i=1}^N U(x_i) : x_i \in \overline{\Omega} \right\},$$

where  $h_N$  is a suitable chosen normalization factor. Since the seminal work of Choquet in 1958 [9] and the growing interest of the quantum and statistical mechanics community, a lot of work has been devoted to the limit behavior of  $\mathcal{E}_N^\varepsilon(\Omega, U)$  as  $N \rightarrow \infty$  ( $\varepsilon$  fixed) as well as the characterization of the weak cluster points of the empirical measures associated with  $N$ -point configurations of minimal energy. The cornerstone of the mean field theory consists in identifying a limit energy functional on measures whose minimizers are precisely these cluster points.

**1.1. State of the art.** The scaling factor  $h_N$  in (1.2) must be selected so that the limit of the infimum belongs to  $(0, +\infty)$ . In turn this issue relies heavily on the integrability properties of the function  $g(x) = \ell(|x|)$ . Let us report on two cases of major interest:

**1.1.1. Long range interaction case.** Here  $\Omega = \mathbb{R}^d$  and we take  $\varepsilon = 1$ . Moreover in addition to (H1)(H2), we assume that  $g \in L_{loc}^1(\mathbb{R}^d)$  i.e.:

$$(1.3) \quad \int_0^1 r^{d-1} \ell(r) dr < +\infty.$$

In that case, a relevant choice is  $h_N = \frac{1}{N^2}$  meaning roughly that the interaction energy is averaged over all pairs of distinct points in  $\{x_1, x_2, \dots, x_N\}$ . The identification of the mean-field energy is well known in the case of Riesz potentials  $\ell(r) = \frac{1}{r^s}$  for  $0 < s < d$ , in the Logarithmic case  $\ell(r) = -\log(r)$  for  $d = 2$  and more generally for  $\ell$  of positive type i.e. such that the Fourier transform of  $g(x) = \ell(|x|)$  is positive in  $\mathbb{R}^d$  (see for instance the monograph by S. Serfaty [20]). It is given by a non-local functional, the so called *Direct energy*:

$$(1.4) \quad D_\ell(\rho) := \iint \ell(|x - y|) \rho \otimes \rho(dx dy).$$

Accordingly the limit problem associated with (1.2) reads:

$$\mathcal{E}_\infty(\Omega, U) = \inf \left\{ D_\ell(\rho) + \int U d\rho : \rho \in \mathcal{P}(\mathbb{R}^d) \right\},$$

where the infimum is reached at a unique configuration provided  $U$  grows suitably at infinity. At this stage, a few comments are in order:

- in the case of a confining external potential  $U$ , there are several impressive works devoted to the next order asymptotics [20, 19, 21, 10] in the case of Riesz potentials  $\ell(r) = r^{-s}$  for  $d \geq 3$  and  $d - 2 \leq s < d$  revealing an asymptotic behavior as  $N \rightarrow \infty$  of the form:

$$(1.5) \quad \lim_{N \rightarrow \infty} N^{1-\frac{s}{d}} (\mathcal{E}_N(\Omega, U) - \mathcal{E}_\infty(\Omega, U)) = C(s, d) \int (\rho_U)^{1+\frac{s}{d}},$$

where  $\rho_U$  is the unique minimizer realizing  $\mathcal{E}_\infty(\Omega, U)$ .

- if  $U$  remains bounded at infinity (for instance a Coulomb potential vanishing at infinity), the existence of a minimizer  $\rho_U$  may fail due to a loss of mass at infinity along minimizing sequences. A relaxation procedure leads to consider minimizers in the class of sub-probabilities  $\rho \in \mathcal{P}_-(\mathbb{R}^d)$  and involves the weak\* lower semicontinuous convexification of the Direct energy  $D$ . If  $\ell$  is of positive type, this relaxed energy coincides with the natural 2-homogeneous extension of  $D$  to  $\mathcal{P}_-(\mathbb{R}^d)$  while almost nothing is known if  $\ell$  is merely locally integrable. For further details and examples of relaxed minimizers, we refer to the recent paper [3].

1.1.2. *Short range interaction case.* Following an idea developed for the hard spheres model [3], we look now at  $\varepsilon$  as a small parameter tending to zero with a prescribed speed as  $N \rightarrow \infty$ . If one thinks to a container  $\bar{\Omega}$  of unit volume and  $\varepsilon$  to be the average distance of a particle to the others, it is natural to consider an asymptotic analysis where the product  $\varepsilon^d N$  remains constant or converges to a given intensity factor  $\kappa \in (0, +\infty)$ . In a crowd model, this factor  $\kappa$  is related to a congestion ratio (see the hard spheres model in Section 3.4 and Remark 3.13). This of course means that we need to assume that  $\varepsilon \sim N^{-\frac{1}{d}}$ . Accordingly, in order to obtain a precise scaling for  $h_N$  ensuring a non-trivial behavior of the infimum (1.2), it is crucial to make an additional integrability on  $\ell$  at infinity namely

$$(H3) \quad \int_{r_0}^{+\infty} \ell(r) r^{d-1} dr < +\infty.$$

It turns out that, under (H3), the right scaling factor in (1.2) is  $h_N = \frac{1}{N}$  in contrast with the long range case. This covers the case of hyper singular Riesz potentials  $\ell(r) = r^{-s}$  with  $s > d$ . For such potentials the parameter  $\varepsilon$  can be dropped thanks to the homogeneity and the normalized interaction energy becomes  $\frac{1}{N^{1+\frac{s}{d}}} c_N^1(x_1, \dots, x_N)$ . Under the latter scaling, it was proved recently [12, 13] that the mean field energy is a local functional defined on absolutely continuous measures  $\rho = u \mathcal{L}^d \llcorner \Omega$  by  $F(\rho) = C(s, d) \int_\Omega u^{1+\frac{s}{d}} d\mathcal{L}^d$ , being  $C(s, d)$  a universal constant. However, extending this result to more general costs seems to be difficult in the framework developed in [12], except possibly if  $\ell$  is very close to a power potential.

**1.2. Our contribution.** This paper proposes a significant simplification of asymptotic analysis in the short range case. The approach is based on two components: first, treating the *interaction distance*  $\varepsilon$  in (1.1) as an infinitesimal parameter, and second, using an  $\varepsilon$ -counterpart of the traditional empirical measure frequently utilized in mean-field theory.

Thus, for every cost  $\ell$  satisfying (H3), we can determine the mean field energy in terms of a *local* integral functional of the type  $\int_{\Omega} f_{\ell}(u) dx$ , where  $u = \frac{d\rho}{dx}$  denotes the local particle density and  $f_{\ell}$  is a convex integrand that exhibits super-linear growth at infinity. This expands upon previous findings [3] that were limited to the hard-spheres model (where  $\ell(r) = +\infty$  when  $r < 1$  and  $\ell(r) = 0$  otherwise). Similarly, this permits to handle the case of hyper-singular Riesz potentials  $\ell(r) = r^{-s}$  for  $s > d$  analyzed in [12, 13].

It is noteworthy that fulfilling the integrability condition (H3) is crucial and cannot be sidestepped. When assuming that  $\Omega$  is bounded, a cost  $\ell$  that satisfies  $\int_{r_0}^{+\infty} r^{d-1} \ell(r) dr = +\infty$  would give rise to an infinite limit in (1.2) if the scaling is by  $h_N = \frac{1}{N}$  and  $\varepsilon^d N \sim 1$  (see Remark 3.14).

**1.3. Setting of the asymptotic problem and notations.** From now on,  $\Omega$  will be a *bounded* domain of  $\mathbb{R}^d$  with Lipschitz boundary ( $\partial\Omega$  needs to be  $\mathcal{L}^d$ -negligible) and we consider a cost function  $\ell$  which satisfies the standing assumptions (H1), (H2) and (H3).

For purposes of presentation, we utilize the infinitesimal length  $\varepsilon$  as the main parameter while the number of particles  $N = N_{\varepsilon}$  approaches infinity as  $\varepsilon$  tends towards zero, following the scale  $N_{\varepsilon} \sim \kappa \varepsilon^{-d}$  where  $\kappa$  is a positive constant. Later, we will establish that assigning a value to  $\kappa$  is not required because a uniform bound on the  $N_{\varepsilon}$ -point interaction energy will automatically result in  $\limsup_{\varepsilon \rightarrow 0} N_{\varepsilon} \varepsilon^d < +\infty$ . We can now

incorporate measures in  $\overline{\Omega}$  into a variational framework for addressing the mean field problem. For each finite subset  $S \subset \overline{\Omega}$ , we define its  $\varepsilon$ -scaled empirical measure as follows:

$$(1.6) \quad \rho_S^{\varepsilon} := \varepsilon^d \sum_{x \in S} \delta_x.$$

This measure belongs to the set of non-negative Borel measures on  $\overline{\Omega}$ , denoted by  $\mathcal{M}_+(\overline{\Omega})$ . Here,  $\|\rho\|$  represents the total mass, which may be infinite, of any element  $\rho \in \mathcal{M}_+(\overline{\Omega})$ . Through this, we can observe that  $\|\rho_S^{\varepsilon}\| = \varepsilon^d \#(S)$ , which could deviate from the classical empirical measure of  $S_{\varepsilon}$  with a total mass equal to 1. A key advantage of this approach is that it allows to avoid the non-local constraint that all competitors must belong to the subclass  $\mathcal{P}(\overline{\Omega})$  of probability measures.

Next, we define the  $\varepsilon$ -scaled interaction energy of a discrete set  $S \subset \mathbb{R}^d$  as follows:

$$(1.7) \quad \xi_{\ell,\varepsilon}(S) = \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left( \frac{|x-y|}{\varepsilon} \right) \quad \text{where} \quad \Delta := \{(x,x) : x \in \mathbb{R}^d\}.$$

We will refer to the interaction energy corresponding to  $\varepsilon = 1$  as the “ground interaction energy”, denoted by  $\xi_\ell$ . When the cost function  $\ell$  is fixed, we will use  $\xi_\varepsilon$  instead of  $\xi_{\ell,\varepsilon}$ . With this in mind, we can define a scaled energy functional  $F_\varepsilon : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$  for every  $\varepsilon > 0$  in the following way:

$$(1.8) \quad F_\varepsilon(\rho) = \begin{cases} \varepsilon^d \xi_\varepsilon(S) & \text{if } \exists S \subset \overline{\Omega} \text{ such that } \rho = \rho_S^\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

The discrete problem (1.2) for  $h_N = \frac{1}{N}$  and  $N \varepsilon^d \sim \kappa$  can then be expressed through the relation:

$$\kappa \mathcal{E}_N^\varepsilon(\Omega, U) \sim \inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho : \rho \in \mathcal{M}_+(\overline{\Omega}) \right\} \quad \text{as } \varepsilon \rightarrow 0.$$

Accordingly the mean-field energy will be represented by a functional  $F : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$  characterized by the property that, for every  $U \in \mathcal{C}(\overline{\Omega})$ , one has the convergence of infima

$$\inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho \right\} \rightarrow \inf \left\{ F(\rho) + \int_{\overline{\Omega}} U d\rho \right\}$$

accompanied by the tight convergence of minimizers. This falls squarely within the  $\Gamma$ -convergence theory ([2, 11, 7]) on which we rely to support our results.

The paper is organized as follows: in Section 2, we establish a lower bound which allows to obtain the strong equi-coercivity of the sequence  $(F_\varepsilon)$ ; in addition we show that any weak\* cluster point of a sequence  $(\rho_\varepsilon)$  with uniformly bounded energy is absolutely continuous with respect to the Lebesgue measure; in Section 3, we state the  $\Gamma$ -convergence of  $F_\varepsilon$  as  $\varepsilon \rightarrow 0$  to a convex functional of the form  $F(\rho) = \int_{\overline{\Omega}} f_\ell \left( \frac{d\rho}{dx} \right) dx$ , where the effective integrand  $f_\ell$  grows at least quadratically at infinity. It is given by the thermodynamical limit of a subadditive set function (*Krengel's theorem*). Some examples and applications are given. The Section 4 is devoted to the proof of the main theorem.

#### Notations:

- $B(x, r)$  is the open ball of the Euclidean space  $\mathbb{R}^d$  centered at  $x$  and of radius  $r$ ; if  $x = 0$ , we simply denote  $B_r$ ;
- $Q_k$  denotes the hypercube  $[-k/2, k/2]^d$ ,  $Q(x_0, r) := x_0 + r Q_1$ ;
- $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$  stands for the diagonal of  $\mathbb{R}^d$ ;

- $\sharp(S)$  denotes the counting measure of a subset  $S \subset \mathbb{R}^d$  ( $+\infty$  if  $S$  is infinite);
- $\mathcal{L}^d$  is the Lebesgue measure in  $\mathbb{R}^d$ ; given any Borel set  $B$ ,  $|B|$  is a short notation for  $\mathcal{L}^d(B)$ ;  $\omega_d$  is such that  $|B(x, r)| = \omega_d r^d$ ;
- $\mathcal{C}(\overline{\Omega})$  denotes the Banach space of continuous functions on the compact subset  $\overline{\Omega}$  equipped with the uniform norm;
- $\mathcal{M}(\overline{\Omega})$  stands for the space of signed Radon measures on  $\overline{\Omega}$  equipped with the total variation norm;
- $\mathcal{P}_-(\overline{\Omega})$  (resp.  $\mathcal{P}(\overline{\Omega})$ ) is the subset of Borel measures  $\mu \in \mathcal{M}_+(\overline{\Omega})$  such that  $\|\mu\| := \mu(\overline{\Omega}) \leq 1$  (resp.  $\|\mu\| = 1$ ).
- The topological support of  $\mu \in \mathcal{M}_+(\overline{\Omega})$  is denoted  $\text{supp}(\mu)$  while  $\mu \llcorner A$  represents its restriction to a Borel subset  $A \subset \overline{\Omega}$ ;
- The bracket  $\langle \cdot, \cdot \rangle$  will denote the duality between  $\mathcal{C}(\overline{\Omega})$  and  $\mathcal{M}(\overline{\Omega})$ :

$$\langle v, \mu \rangle = \int v d\mu,$$

This duality induces the weak\* topology on  $\mathcal{M}(\overline{\Omega})$  which can be identified with the dual of  $\mathcal{C}(\overline{\Omega})$ ; as  $\overline{\Omega}$  is compact, the weak\* convergence  $\mu_h \xrightarrow{*} \mu$  in  $\mathcal{M}_+(\overline{\Omega})$  implies the tight convergence since  $\|\mu_h\| = \langle 1, \mu_h \rangle \rightarrow \langle 1, \mu \rangle = \|\mu\|$ .

- To any non-empty set  $A$ , we associate the functions:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}, \quad \chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}.$$

## 2. ENERGY ESTIMATES AND COMPACTNESS.

We begin with some elementary properties of the set function  $\xi_\ell(S)$  (defined by (1.7) for  $\varepsilon = 1$ ).

**Lemma 2.1.** *Let  $S_1, S_2$  be finite disjoint subsets of  $\mathbb{R}^d$ . Then we have:*

(i) (super-additivity)

$$\xi_\ell(S_1 \cup S_2) \geq \xi_\ell(S_1) + \xi_\ell(S_2).$$

(i) (sub-additivity at large distance)

$$\xi_\ell(S_1 \cup S_2) \leq \xi_\ell(S_1) + \xi_\ell(S_2) + 2\ell_+(\eta) \sharp(S_1) \sharp(S_2),$$

where  $\eta := \text{dist}(S_1, S_2)$ .

*Proof.* Since  $S_1$  and  $S_2$  are non-intersecting, we can split  $(S_1 \cup S_2)^2$  in four disjoint pieces as follows:

$$(S_1 \cup S_2)^2 = (S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_1 \times S_2) \cup (S_2 \times S_1).$$

The inequality (i) is then straightforward whereas, for the (ii), we simply majorize by  $\ell_+(\eta)$  the contribution  $\ell(|x - y|)$  of each pair  $(x, y)$  in  $(S_1 \times S_2) \cup (S_2 \times S_1)$  where  $|x - y| \geq \eta$  holds.  $\square$

Recalling the notations (1.6) and (1.8) given in the introduction, we may rewrite a  $\varepsilon$ - rescaled version of Lemma 2.1 as follows: for every  $\varepsilon > 0$  and all pairs of non-intersecting subsets  $(S'_\varepsilon, S''_\varepsilon)$ , one has

$$(2.1) \quad F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon + \rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) + \frac{2\ell_+(\eta_\varepsilon)}{\varepsilon^d} \|\rho'_\varepsilon\| \|\rho''_\varepsilon\|,$$

where  $\rho'_\varepsilon := \rho_{S'_\varepsilon}^\varepsilon$ ,  $\rho''_\varepsilon := \rho_{S''_\varepsilon}^\varepsilon$  and  $\eta_\varepsilon = \varepsilon^{-1} \text{dist}(S'_\varepsilon, S''_\varepsilon)$ .

**2.1. A fundamental lower-bound.** Since we are only assuming that  $\ell(r)$  is decreasing for suitably large  $r$ , we need to define :

$$(2.2) \quad \ell_+(r) := \sup \{ \ell(s) : s \geq r \} \quad , \quad \ell_-(r) := \inf_{x,y \in [0,r]^d} \ell(|x-y|)$$

which are monotone non-increasing and satisfy:

$$\ell_+(r) = \ell(r) \quad \forall r \geq r_0 \quad , \quad \ell_-(r) = \inf \{ \ell(s) : s \leq r\sqrt{d} \} \leq \ell_+(r\sqrt{d}).$$

Next, in the same line as in the survey [17], we derive a very simple but fundamental lower bound for  $\xi_\ell(S)$  when  $S$  is a  $N$ -point system contained in a Borel subset  $B$  of finite volume in  $\mathbb{R}^d$ . For such a  $B$  and any  $\delta > 0$ , we denote by  $m_\delta(B)$  the minimal number of disjoint  $\delta$ -hypercubes  $Q_j = x_j + [-\delta/2, \delta/2]^d$  such that  $B \subset \cup_{j=1}^{m_\delta(B)} \overline{Q_j}$ . Then it's easy to check that  $m_\delta(B) \sim \delta^{-d} \mathcal{L}^d(B)$  as  $\delta \rightarrow 0$ . For such subsets  $B$ , we will often use the following equivalent version obtained by keeping  $\delta$  fixed while using large dilations:

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^d m_\delta \left( \frac{B}{\varepsilon} \right) = \delta^{-d} \mathcal{L}^d(B).$$

**Lemma 2.2.** *Let  $B \subset \mathbb{R}^d$  be a Borel subset such that  $\mathcal{L}^d(B) < +\infty$ . Then for any  $N$ -point system  $S \subset B$  and any  $\delta > 0$ , we have*

$$(2.4) \quad \xi_\ell(S) \geq N \ell_-(\delta) (\zeta - 1)_+ \quad \text{where} \quad \zeta = \frac{N}{m_\delta(B)}.$$

*Proof.* Let  $\{Q_j, 1 \leq j \leq m_\delta(B)\}$  be a covering of  $B$  by disjoint  $\delta$ -hypercubes  $Q_j$  and denote  $n_j = \#(S \cap Q_j)$ . We have  $N = \sum n_j$  while, by the super additivity of  $\xi_\ell$  (see Lemma 2.1) and the definition of  $\ell_-(\delta)$ , we have:

$$\xi_\ell(S) \geq \sum_{j=1}^{m_\delta(B)} \xi_\ell(S \cap Q_j) \geq \ell_-(\delta) \sum_{j=1}^{m_\delta(B)} n_j(n_j - 1).$$

If  $N > m_\delta(B)$  (i.e.  $\zeta > 1$ ), the desired lower bound (2.4) follows by noticing that the infimum

$$\inf \left\{ \sum_{j=1}^{m_\delta(B)} t_j(t_j - 1) : \sum_{j=1}^{m_\delta(B)} t_j = N, t_j \in \mathbb{R} \right\}$$



is reached for  $t_j = \frac{N}{m_\delta(B)} = \zeta, \forall j$ .

If  $\zeta \leq 1$  the inequality (2.4) is trivial since  $\xi_\ell(S) \geq 0$ .  $\square$

**Remark 2.3.** By the lower semicontinuity assumption (H1), since  $\ell_-(0) = \ell(0) \in (0, +\infty]$ , we may always find a  $\delta > 0$  such that  $\ell_-(\delta) > 0$ . Note that the inequality (2.4) is still valid if  $\ell_-(\delta) = +\infty$  provided we agree that  $0 \times +\infty = 0$  (this situation occurs when  $\ell = +\infty$  on an interval  $[0, \delta_0)$ ).

**2.2. Strong coercivity and compactness.** We recall the definition of the functional  $F_\varepsilon$  given in (1.8) whose domain consists of  $\varepsilon$ -empirical measures of discrete subsets  $S_\varepsilon \subset \overline{\Omega}$  (see the definition (1.6)).

**Lemma 2.4** (strong coercivity). *Assuming that  $\ell$  satisfies (H1) (H2), let  $\delta > 0$  be such that  $\ell_-(\delta) \in (0, +\infty]$  and set  $\alpha := \frac{\delta^d \ell_-(\delta)}{\mathcal{L}^d(\Omega)}$ .*

*Then we have:*

$$(2.5) \quad \liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|^2} \geq \alpha,$$

*whenever  $(\rho_\varepsilon)$  is a sequence such that  $\|\rho_\varepsilon\| \rightarrow +\infty$ .*

*Proof.* Without loss of generality, we may assume that  $F_\varepsilon(\rho_\varepsilon) < +\infty$ . Thus  $\rho_\varepsilon = \rho_{S_\varepsilon}^\varepsilon$  for a suitable  $N_\varepsilon$ -point system  $S_\varepsilon \subset \overline{\Omega}$  while:

$$F_\varepsilon(\rho) = \varepsilon^d \xi_\varepsilon(S_\varepsilon) = \varepsilon^d \xi_\ell \left( \frac{S_\varepsilon}{\varepsilon} \right), \quad \|\rho_\varepsilon\| = N_\varepsilon \varepsilon^d.$$

By applying the lower bound (2.4) to the subset  $\varepsilon^{-1}S \subset \varepsilon^{-1}\overline{\Omega}$ , we get:

$$(2.6) \quad F_\varepsilon(\rho_\varepsilon) \geq \ell_-(\delta) \|\rho_\varepsilon\| \left( \frac{\|\rho_\varepsilon\|}{\beta_\varepsilon} - 1 \right)_+ \quad \text{where } \beta_\varepsilon = \varepsilon^d m_\delta \left( \frac{\overline{\Omega}}{\varepsilon} \right).$$

From (2.3), we know that  $\beta_\varepsilon \rightarrow \delta^{-d} \mathcal{L}^d(\Omega)$ . Hence (2.5) follows by dividing (2.6) by  $\|\rho_\varepsilon\|^2$  and passing to the limit as  $\varepsilon \rightarrow 0$ .  $\square$

**Proposition 2.5** (compactness). *Assuming that  $\ell$  satisfies (H1) and (H2), let  $U : \overline{\Omega} \rightarrow \mathbb{R}$  be a bounded Borel function and  $(\rho_\varepsilon)$  a sequence in  $\mathcal{M}_+(\overline{\Omega})$  such that*

$$(2.7) \quad \sup_\varepsilon \left( F_\varepsilon(\rho_\varepsilon) + \int U d\rho_\varepsilon \right) < +\infty.$$

*Then :*

(i) *there exists a constant  $C$  such that*

$$\|\rho_\varepsilon\| + F_\varepsilon(\rho_\varepsilon) \leq C < +\infty \quad \text{for every } \varepsilon > 0.$$

(ii) *any weak\* cluster point of  $(\rho_\varepsilon)$  is of the form  $\rho = u \mathcal{L}^d \llcorner \Omega$  with  $u \in L^1(\Omega)$ .*

**Remark 2.6.** The proposition above implies that sequences of point configurations  $S_\varepsilon$  with equi-bounded energies admit a finite limiting intensity factor  $\kappa := \limsup_\varepsilon \varepsilon^d \sharp(S_\varepsilon)$  while, by the assertion (ii),  $S_\varepsilon$  is not allowed to concentrate anywhere as  $\varepsilon \rightarrow 0$ .

*Proof.* Suppose that  $\|\rho_\varepsilon\|$  has no upper bound. Then the uniform energy upper bound (2.7) implies that

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|} \leq \sup_{\bar{\Omega}} |U|,$$

while by (2.5) the left hand member of the previous inequality is infinite. So there is a contradiction and we can conclude that  $(\rho_\varepsilon)$  is bounded. Then it follows from (2.7) that  $F_\varepsilon(\rho_\varepsilon)$  is bounded as well, whence the assertion (i).

Let us now prove the assertion (ii); we know that the sequence  $(\rho_\varepsilon)$  is bounded and therefore admits weak\* cluster points. Given such a cluster point, we can assume, without loss of generality, that  $\rho_\varepsilon \xrightarrow{*} \rho$  in  $\mathcal{M}_+(\bar{\Omega})$ . Let us introduce for every  $t > 0$  the set

$$E_t := \left\{ x \in \bar{\Omega} : \liminf_{r \rightarrow 0} \frac{\rho(B(x, r))}{\omega_d r^d} > t \right\}.$$

Thanks to the upper semi-continuity of the map  $x \rightarrow \rho(B(x, r))$ , we infer that  $E_t$  is a Borel subset of  $\bar{\Omega}$ . We are going to prove that

$$(2.8) \quad \lim_{t \rightarrow +\infty} \rho(E_t) = 0.$$

To that aim, we consider the family of closed balls in  $R^d$  defined by

$$\mathcal{F}_t := \bigcup \left\{ \overline{B(x, r)} : x \in E_t, r < r_x, \rho(\partial B(x, r)) = 0 \right\},$$

where  $r_x > 0$  is chosen so that  $\rho(B(x, r)) > t \omega_d r^d$  for every  $r < r_x$ . Since  $\mathcal{F}_t$  determines a fine covering of the bounded Borel set  $E_t$ , we may invoke the Vitali-Besicovitch covering theorem (see [1, Thm 2.19]) which provides the existence a countable subfamily  $(B_n)$  such that

$$(2.9) \quad \rho(B_n) > t \mathcal{L}^d(B_n) \quad \forall n \quad , \quad \rho(E_t \setminus \cup_n B_n) = 0.$$

Next we associate with the weak\* convergent sequence  $(\rho_\varepsilon)$ , two set functions defined on Borel subsets  $A \subset \bar{\Omega}$ :

$$\eta_\varepsilon(A) := F_\varepsilon(\rho_\varepsilon \llcorner A) \quad , \quad \eta(A) := \liminf_{\varepsilon \rightarrow 0} \eta_\varepsilon(A).$$

We can readily check that  $\eta_\varepsilon$  and  $\eta$  are monotone with respect to the inclusion while  $\eta_\varepsilon(\bar{\Omega}) = F_\varepsilon(\rho_\varepsilon)$  implies that  $\eta(\bar{\Omega}) \leq \beta < +\infty$ . Moreover, by the first inequality in (2.1),  $\eta_\varepsilon$  is super-additive on disjoint Borel subsets. Obviously this holds true also for the set function  $\eta$ . By

applying this property to the sequence of disjoint balls  $B_n$ , we get the upperbound:

$$(2.10) \quad \sum_n \eta(B_n) \leq C .$$

On the other hand, thanks to the coercivity inequality (2.6) that we apply with  $\bar{\Omega} = B_n$ , we obtain:

$$(2.11) \quad \eta_\varepsilon(B_n) = F_\varepsilon(\rho_\varepsilon \llcorner B_n) \geq \ell_-(\delta) \rho_\varepsilon(B_n) \left( \frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} - 1 \right)_+ ,$$

where  $\beta_\varepsilon(B_n) := \varepsilon^d m_\delta \left( \frac{B_n}{\varepsilon} \right)$ . Since  $\rho(\partial B_n) = 0$  by construction and thanks to (2.3) and (2.9), we infer that

$$\lim_{\varepsilon \rightarrow 0} \frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} = \frac{\rho(B_n)}{\delta^{-d} \mathcal{L}^d(\mathcal{B}_n)} \geq t \delta^d .$$

Therefore, passing to the limit  $\varepsilon \rightarrow 0$  in (2.11), we deduce that:

$$(2.12) \quad \eta(B_n) \geq \ell_-(\delta) \rho(B_n) (t \delta^d - 1)_+ .$$

All in all, after collecting the second equality of (2.9), (2.10) and (2.12), we are led to:

$$\rho(E_t) \leq \sum_n \rho(B_n) \leq \frac{C}{\ell_-(\delta)(t \delta^d - 1)_+} .$$

Our claim (2.8) follows by sending  $t \rightarrow +\infty$ . The absolute continuity property  $\rho \ll \mathcal{L}^d$  stated in the assertion (ii) is a consequence of the *Besicovitch differentiation theorem* [1, theorem 2.22]), which states that the singular part  $\rho_s$  in the Lebesgue-Nikodym decomposition of  $\rho$  with respect to the Lebesgue measure coincides with  $\rho \llcorner E_\infty$  being  $E_\infty = \bigcap_{t>0} E_t$ . In our case  $\|\rho_s\| = \rho(E_\infty) = 0$  due to (2.8).  $\square$

**2.3. Upper-bound of energies.** In the same way as in [17], we will be using an upper bound of  $\xi_\ell(S)$  when  $S$  is arranged on a  $d$ -dimensional periodic Bravais lattice  $\mathcal{G}$ <sup>1</sup>. To such a lattice we associate the  $\ell$ -*Epstein zeta function* defined for every  $r > 0$  by:

$$(2.13) \quad \Lambda_{\ell, \mathcal{G}}(r) := \sum_{x \in \mathcal{G} \setminus \{0\}} \ell(r|x|) .$$

In the case where the cartesian lattice  $\mathcal{G} = \mathbb{Z}^d$  is used, we will write simply  $\Lambda_\ell(r)$ . The finiteness of this function for large  $r$ , under the condition (H3), turns out to be crucial for deriving an uniform upper bound for the scaled energy  $F_\varepsilon$  given in (1.8).

<sup>1</sup>i.e. of the form  $\mathcal{G} = F \mathbb{Z}^d$  for some invertible matrix  $F \in \mathbb{R}^{d \times d}$

**Lemma 2.7.** *Under (H1) – (H3), there exists  $C_{\mathcal{G}} > 0$  such that*

$$(2.14) \quad \Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \left( \ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt \right) \quad \forall r \geq r_0 \max\{1, a_{\mathcal{G}}^{-1}\},$$

where  $a_{\mathcal{G}} := \min\{|y| : y \in \mathcal{G} \setminus \{0\}\}$ .

*Proof.* To simplify, we chose the lattice  $\mathcal{G}$  so that  $a_{\mathcal{G}} = 1$ . Up to substituting  $\ell$  with  $\ell_+$  which satisfies  $\ell_+ \geq \ell$  and  $\ell_+ = \ell$  on  $[r_0, +\infty)$ , we may also assume that  $\ell$  is non-increasing on  $\mathbb{R}_+$ . Accordingly, for any  $s \in [0, \ell(0_+))$ , the set of values  $\{\ell > s\}$  forms a non-empty interval  $[0, \ell^{-1}(s))$ . The pseudo inverse  $\ell^{-1}(s)$  is the supremum of all  $t \geq 0$  such that  $\ell(t) > s$ , and it is a monotone non-increasing function on  $[0, +\infty)$ . Therefore, we have the following equivalence:

$$\ell^{-1}(s) > t \iff \ell(t) > s.$$

By applying the layer cake formula to the counting measure on  $\mathcal{G}$ , we get

$$\Lambda_{\ell, \mathcal{G}}(r) = \int_0^{\infty} N_r(s) ds,$$

where the integer function  $N_r(s) := \#\{\{x \in \mathcal{G} \setminus \{0\} : \ell(r|x|) > s\}\}$  satisfies  $N_r(s) = 0$  if  $r \geq r_0$  and  $s \geq \ell(r_0)$  (we assumed that  $a_{\mathcal{G}} = 1$ ). On the other hand, for any periodic Bravais lattice  $\mathcal{G} \subset \mathbb{R}^d$ , there exists a constant  $C_{\mathcal{G}} > 0$  such that

$$\#\{B_r \cap \mathcal{G}\} \leq C_{\mathcal{G}} r^d, \quad \forall r > 0.$$

This implies the inequality:

$$N_r(s) = \#\{\{x \in \mathcal{G} \setminus \{0\} : r|x| < \ell^{-1}(s)\}\} \leq \frac{C_{\mathcal{G}}}{r^d} (\ell^{-1}(s))^d.$$

Therefore, for every  $r \geq r_0$ , we are led to:

$$\Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds.$$

Then, after noticing that  $\{s \in [0, \ell(r_0)] : \ell^{-1}(s) > t\} = [0, \ell(r_0) \wedge \ell(t)]$  holds for any  $t \geq 0$ , we obtain the desired inequality by applying once again the layer cake formula:

$$\begin{aligned} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds &= d \int_0^{\infty} (\ell(r_0) \wedge \ell(t)) t^{d-1} dt \\ &= \ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt. \end{aligned}$$

□

**Remark 2.8.** In view of Lemma 2.7, the  $\ell$ - Epstein zeta function  $\Lambda_{\ell, \mathcal{G}}(r)$  vanishes at infinity. However the behavior in  $O(r^{-d})$  as  $r \rightarrow \infty$  suggested by (2.14) is not optimal as we can see in the case of a Riesz potential  $\ell(r) = r^{-s}$  with  $s > d$ , where  $\Lambda_{\ell, \mathcal{G}}(r) = C r^{-s}$ . On the other

hand, it is noteworthy that  $\Lambda_{\ell, \mathcal{G}}$  is not continuous in general. A very simple example to see this is given by the step function  $\ell = \frac{1}{2} \mathbf{1}_{[0,1]}$  which satisfies (H1) – (H3). For  $d = 1$  and the lattice  $\mathcal{G} = \mathbb{Z}$ , we find that

$$\Lambda_{\ell}(r) = \#\{n \in \mathbb{N} : 0 < nr < 1\} = [r^{-1}] ,$$

where  $[\cdot]$  denotes the integer part.

Next, by applying Lemma 2.7 in the case of the Cartesian lattice  $\mathcal{G} = \mathbb{Z}^d$ , we derive a fundamental upper bound for the short-range interaction energy.

**Lemma 2.9.** *Let  $r > 0$  and  $S$  be a finite subset of the lattice  $r\mathbb{Z}^d$ . Then*

$$(2.15) \quad \xi_{\ell}(S) \leq \#(S) \Lambda_{\ell}(r)$$

As a consequence, for every  $a > 0$ , there exists  $S_{\varepsilon} \subset \bar{\Omega}$  such that  $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$  satisfies

$$(2.16) \quad \rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega \quad \text{and} \quad \limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|$$

where the right hand side upper bound is finite whenever  $0 \leq a \leq r_0^{-d}$ .

*Proof.* Let  $S = \{r x_i : 1 \leq i \leq N\}$  where  $N = \#(S)$  and  $x_i \in \mathbb{Z}^d$ . Noticing that, for every  $i$ , the set  $\{x_i - x_j : j \neq i\}$  consists of  $N - 1$  distinct elements of  $\mathbb{Z}^d \setminus \{0\}$ , we infer that

$$\sum_{j \neq i} \ell(r|x_i - x_j|) \leq \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \ell(r|z|) = \Lambda_{\ell}(r),$$

hence the desired inequality (2.15) by summing with respect to  $i$ .

Taking now  $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$  where  $S_{\varepsilon} = \Omega \cap (r_{\varepsilon}\mathbb{Z})^d$  and  $r_{\varepsilon} = \varepsilon a^{-\frac{1}{d}}$ , we obtain a sequence such that  $\rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega$  as  $\varepsilon \rightarrow 0$ . Indeed, by the periodicity of the Euclidean lattice,  $\rho_{\varepsilon}$  converges to a uniform density on  $\Omega$  while its total mass  $\|\rho_{\varepsilon}\| = \varepsilon^d \#(S_{\varepsilon}) \sim \varepsilon^d (r_{\varepsilon}^{-d} |\Omega|)$  converges to  $a |\Omega|$  as  $\varepsilon \rightarrow 0$ . Eventually, by applying (2.15), we get  $F_{\varepsilon}(\rho_{\varepsilon}) = \varepsilon^d \xi_{\ell}(\frac{S_{\varepsilon}}{\varepsilon}) \leq \varepsilon^d N_{\varepsilon} \Lambda_{\ell}(a^{-1/d})$ , whence:

$$\limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|.$$

The finiteness of  $\Lambda_{\ell}(a^{-\frac{1}{d}})$  for  $a \in [0, r_0^{-d}]$  follows from Lemma 2.7.  $\square$

**Remark 2.10.** The upper bound (2.15) and (2.16) obtained by choosing  $\mathcal{G} = \mathbb{Z}^d$  as the reference lattice are in general not optimal since the Epstein-zeta function of another lattice could provide better ones. Obviously the inequality (2.15) holds true after replacing  $\Lambda_{\ell}$  with the Epstein function  $\Lambda_{\ell, \mathcal{G}}$  of any Bravais lattice  $\mathcal{G} = F\mathbb{Z}^d$ , while for the validity of (2.15), we need to add the normalization condition  $|\det(F)| = 1$

(thereby fixing the volume of the so called *fundamental domain* of  $\mathcal{G}$ ). The existence and the determination of an optimal lattice  $\mathcal{G}$  for the following minimization problem:

$$\inf\{\Lambda_{\ell,\mathcal{G}}(r) : \mathcal{G} = F Z^d, |\det(F)| = 1\}$$

touches on a very hard and famous problem related to crystallisation conjectures (see [17, 4]). Note that, for a general cost  $\ell$ , the answer to this problem will depend of the value  $r$  (thus of the local density  $a = \frac{d\rho}{dx}$  of the limiting measure  $\rho$ ).

### 3. THE $\Gamma$ -CONVERGENCE RESULT.

**3.1. A quick overlook.** The notion of  $\Gamma$ -convergence is popular in the community of calculus of variations and very much used in the analysis of sharp-interface models, dimension reduction for problems in mechanics, optimal design and homogenization. As pointed out in the introduction, this tool is also perfectly suited to justify a mean-field approach for large particle systems subject to a minimum energy criterion (see [21]). For the convenience of the reader, let us give here some basic definitions and main properties. For further details, we refer to the monographs [2, 11, 7].

Let  $(E, \tau)$  be a metrizable topological space and consider a sequence of functionals  $F_n : E \rightarrow (-\infty, +\infty]$ . Then the lower  $\Gamma$ -limit  $F_-$  and the upper  $\Gamma$ -limit  $F_+$  of  $F_n$  are defined by:

$$F_-(u) := \inf_{u_n \rightarrow u} \liminf_{n \rightarrow \infty} F_n(u_n) \quad , \quad F_+(u) := \inf_{u_n \rightarrow u} \limsup_{n \rightarrow \infty} F_n(u_n).$$

Both are  $\tau$ -lower semicontinuous (see [2]), whereas in general it holds  $F_- \leq F_+$ . In practice it is useful to check that these functionals are proper i.e. that they range into  $\mathbb{R} \cup \{+\infty\}$  being not identically  $+\infty$ . If  $F_n$  admits a lower bound independent of  $n$ , this amounts to checking that the existence of  $u_0 \in E$  such that

$$(3.1) \quad F_+(u_0) = \inf_{u_n \rightarrow u_0} \limsup_{n \rightarrow \infty} F_n(u_n) < +\infty.$$

We say that  $F_n$   $\Gamma$ -converges to  $F$  (denoted  $F_n \xrightarrow{\Gamma} F$ ) if  $F = F_- = F_+$  or equivalently if the two following conditions are fulfilled:

- a) (*lowerbound*) For any sequence  $u_n$  converging to  $u$ , we have the inequality

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F(u);$$

- b) (*recovering sequence*) For every  $u \in X$  such that  $F(u) < +\infty$ , there exists  $(u_n)$  such that

$$u_n \rightarrow u \quad \text{and} \quad F_n(u_n) \rightarrow F(u).$$

This is the case in particular if  $F_n = F$  does not depend of  $n$ ; then  $F_n \xrightarrow{\Gamma} \text{cl}(F)$  where  $\text{cl}(F)$  denotes the  $\tau$ -lower semicontinuous envelope of  $F$ . Among all properties of  $\Gamma$ -convergence, we give some which will be used in this paper.

**Proposition 3.1.** *Let  $F_n : E \rightarrow (-\infty, +\infty]$  and assume (3.1). Then:*

- (i)  $F_n \xrightarrow{\Gamma} F \iff \text{cl}(F_n) \xrightarrow{\Gamma} F$ ;
- (ii) (Kuratowski compactness Theorem) *If  $(E, \tau)$  is a second countable topological space (for instance a separable metric space), then any sequence  $(F_n)$  admits a  $\Gamma$ -convergent subsequence;*
- (iii) (convergence of infima) *Suppose that  $F_n \xrightarrow{\Gamma} F$  and that the following equi-coercivity property holds:*

$$\sup_n F_n(u_n) < +\infty \implies \{u_n\} \text{ is } \tau\text{-relatively compact.}$$

*Then  $\lim_{n \rightarrow \infty} \inf_X F_n = \min_X F$  and the minimum set for  $F$  coincides with the cluster points of all sequences  $(u_n)$  such that  $F_n(u_n) - \inf F_n \rightarrow 0$  ;*

- (iv) (stability)  $F_n \xrightarrow{\Gamma} F \implies F_n + G \xrightarrow{\Gamma} F + G$  for every continuous perturbation function  $G : E \rightarrow \mathbb{R}$ .

**Remark 3.2.** The continuity requirement for  $G$  in the assertion (iv) is often too restrictive. Actually the same conclusion holds under the following milder condition:

$$(3.2) \quad \begin{cases} \inf_K G > -\infty & \text{for any compact } K \subset E \\ G(u_n) \rightarrow G(u) & \text{whenever } u_n \rightarrow u \text{ and } F(u) < +\infty \end{cases}$$

For the convenience of the reader, a brief proof of the sufficiency of this condition is given below.

*Proof.* To check condition a), we consider a sequence  $(u_n)$  such that  $u_n \rightarrow u$ . Without loss of generality, assume that  $F_n(u_n) + G(u_n) \leq C$  for a suitable constant  $C$ . Since  $G(u_n)$  is lower bounded (take  $K$  to be  $\{u_n, n \in \mathbb{N}\} \cup \{u\}$ ), we infer that  $F_n(u_n) \leq C'$  for another constant  $C'$ . By the  $\Gamma$ -convergence  $F_n \xrightarrow{\Gamma} F$ , it follows that

$$F(u) \leq \liminf_n F_n(u_n) < +\infty.$$

Therefore  $G(u_n) \rightarrow G(u)$  and  $\liminf_n (F_n + G)(u_n) \geq (F + G)(u)$ .

For checking condition b), we may restrict to elements  $u \in E$  such that  $(F + G)(u) < +\infty$ . Then  $F(u) < +\infty$  and any recovering sequence  $u_n \rightarrow u$  such that  $F_n(u_n) \rightarrow F(u)$  will satisfy  $G(u_n) \rightarrow G(u)$ .  $\square$

**3.2. The main result.** In our context, the  $\Gamma$ -convergence issue applies to the sequence  $(F_\varepsilon)$  defined in (1.8) and to the ambient topological space  $\mathcal{M}_+(\overline{\Omega})$  embedded with the weak\* topology (tight convergence). Thanks to the equi-coercivity property established in Proposition 2.5, there is no loss of generality in working in a fixed closed ball of  $\mathcal{M}_+(\overline{\Omega})$  which is metrizable and compact. Therefore all the properties mentioned in the former subsection are applicable (after substituting the index  $n \rightarrow \infty$  with the continuous parameter  $\varepsilon \rightarrow 0$ ).

**Theorem 3.3.** *Let  $\ell$  satisfy the standing assumptions (H1) – (H3) and let  $F_\varepsilon : \mathcal{M}_+(\overline{\Omega}) \mapsto [0, +\infty]$  be given by (1.8). Then  $F_\varepsilon \xrightarrow{\Gamma} F$  (for the weak\* topology) where*

$$F(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

and  $f_\ell : \mathbb{R}_+ \rightarrow [0, +\infty]$  is convex, l.s.c. and satisfies

$$(3.3) \quad f_\ell(0) = f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} > 0.$$

The proof of Theorem 3.3 is postponed to Section 5 while the property (3.3) is established in the next Sub-section. Let us now consider the Fenchel conjugate of  $f_\ell$  (implicitly extended by  $+\infty$  on  $(-\infty, 0)$ ) given by

$$(3.4) \quad f_\ell^*(\lambda) := \sup \{ \lambda t - f_\ell(t) : t \in \mathbb{R}_+ \}$$

From (3.3), one can check that the supremum in (3.4) is actually a maximum (which is attained at  $t = 0$  if  $\lambda \leq 0$ ). Therefore,  $f_\ell^*$  is convex, continuous and vanishes on  $(-\infty, 0]$ . As a consequence it admits left and right derivatives  $(f_\ell^*)'(\lambda_-) \leq (f_\ell^*)'(\lambda_+)$  for any  $\lambda$  so that the subdifferential  $\partial f_\ell^*(\lambda) = [(f_\ell^*)'(\lambda_-), (f_\ell^*)'(\lambda_+)]$  is non empty.

**Corollary 3.4.** *Let  $U \in \mathcal{C}(\overline{\Omega})$  be an external potential. Then, for every  $\varepsilon > 0$ , there exists a finite set  $S_\varepsilon \subset \overline{\Omega}$  minimizing*

$$\mathcal{I}_\ell^{(\varepsilon)}(U) := \inf_{S \subset \overline{\Omega}} \left\{ \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left( \frac{|x-y|}{\varepsilon} \right) + \sum_{x \in S} U(x) \right\}.$$

Moreover  $\sup_\varepsilon \varepsilon^d \sharp(S_\varepsilon) < +\infty$  and  $\lim_{\varepsilon \rightarrow 0} \varepsilon^d \mathcal{I}_\ell^{(\varepsilon)}(U) = \mathcal{I}_\ell(U)$ , where

$$(3.5) \quad \mathcal{I}_\ell(U) := \min_{u \in L^1(\Omega)} \left\{ \int_{\Omega} (f_\ell(u) + uU) dx \right\} = - \int_{\Omega} f_\ell^*(-U) dx.$$

Furthermore any weak\* cluster point of  $\rho_{S_\varepsilon}^\varepsilon$  belongs to the minimum set of (3.5) given by

$$(3.6) \quad \mathcal{S}_\ell := \{ u \mathcal{L}^d \llcorner \Omega : u(x) \in \partial f_\ell^*(-U(x)) \text{ a.e. } x \in \Omega \}.$$



**Remark 3.5** (*Non-uniqueness*). In general  $f_\ell$  is not strictly convex and  $\partial f_\ell^*$  can be multi-valued (see for instance the hard spheres case depicted in Subsection 3.4 or the example of the step function  $\ell$  given in (3.19) where  $f_\ell$  is piecewise affine). Note that, since  $f_\ell^*$  vanishes on  $\mathbb{R}_-$ , it holds  $\mathcal{I}_\ell(U) = 0$  for every non-negative potential  $U$ .

**Remark 3.6.** A natural variant of  $\mathcal{I}_\ell^{(\varepsilon)}(U)$  consists in prescribing the total number  $N_\varepsilon$  of particles to satisfy  $N_\varepsilon \sim \kappa \varepsilon^{-d}$  for some given intensity factor  $\kappa \in (0, +\infty)$ . Accordingly, Corollary 3.4 can be restated by adding a total mass constraint in the limit problem, that is with  $I_\ell(U)$  in (3.5) replaced by

$$\mathcal{I}_{\ell,\kappa}(U) := \inf \left\{ \int_{\Omega} (f_\ell(u) + uU) dx : u \in L^1(\Omega; \mathbb{R}_+) , \int_{\Omega} u dx = \kappa \right\}.$$

The associated minimum  $\mathcal{S}_{\ell,\kappa}$  can be determined by selecting a suitable Lagrange multiplier depending implicitly on  $\kappa$ ; as a consequence, an explicit form for  $\mathcal{S}_{\ell,\kappa}$  of the kind (3.6) is not available.

**Remark 3.7** (*Clustering*). For  $\varepsilon > 0$  fixed, the lower semicontinuity property of  $F_\varepsilon$  requires that  $\ell(0+) = +\infty$ . Otherwise, if  $\ell(0+) < +\infty$ , a sequence of subsets  $S_n \subset \bar{\Omega}$  such that  $\sup_n F_\varepsilon(\rho_{S_n}^\varepsilon) < +\infty$  (thus retaining a finite number of points) can collapse into several clusters while retaining finite energy. In this case, the relaxed functional  $\text{cl}(F_\varepsilon)$  can be obtained directly by extending its domain to  $\varepsilon$ -empirical measures associated with multisets (instead of sets) and by extending the definition (1.6) accordingly taking into account the multiplicity of each cluster of particles. Note however that considering  $\text{cl}(F_\varepsilon)$  instead of  $F_\varepsilon$  will not change the mean-field energy  $F$  given in Theorem 3.3 in virtue of the assertion i) of Proposition 3.1.

**3.3. Characterization and properties of  $f_\ell$ .** The convex integrand  $f_\ell$  will be characterized indirectly through its Fenchel conjugate. For every  $\lambda \in \mathbb{R}$  and any Borel subset  $B \subset \mathbb{R}^d$ , we define:

$$(3.7) \quad \Gamma_\ell(\lambda, B) := \sup \{ \lambda \#(S) - \xi_\ell(S) : S \text{ finite } \subset B \}.$$

The key properties of this bivariate function are summarized in the two following lemmas. Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be the convex, continuous function defined by:

$$(3.8) \quad \varphi(t) = \begin{cases} \frac{(1+t)^2}{4} & \text{if } t \geq 1 \\ t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t < 0 \end{cases}.$$

A straightforward computation shows that its Fenchel conjugate is given by:

$$\varphi^*(\zeta) = \zeta(\zeta - 1)_+ \quad \text{if } \zeta \geq 0 \quad , \quad \varphi^*(\zeta) = +\infty \quad \text{if } \zeta < 0.$$

**Lemma 3.8.** *Assume that  $\mathcal{L}^d(B) < +\infty$ . Then the map  $\lambda \rightarrow \Gamma_\ell(\lambda, B)$  is convex, continuous, vanishes on  $\mathbb{R}_-$  and satisfies:*

$$(3.9) \quad 0 \leq \Gamma_\ell(\lambda, B) \leq m_\delta(B) \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

for every  $\lambda \in \mathbb{R}$ , being  $\delta > 0$  such that  $\ell_-(\delta) \in (0, +\infty]$  and  $\varphi$  being defined by (3.8).

Notice that the right hand side of (3.9) is convex as a function  $\lambda$ . On the other hand, due to the linear behavior of  $\varphi$  on  $[0, 1]$ , we can easily infer that  $\Gamma_\ell(\lambda, B) \leq \lambda m_\delta(B)$  whenever  $\ell_-(\delta) = +\infty$ .

*Proof.* As a supremum of affine functions,  $\Gamma_\ell(\lambda, B)$  is convex and l.s.c. with respect to  $\lambda$ ; it is non-negative (follows by taking  $S$  the empty set) and vanishes if  $\lambda \leq 0$ . The continuity property follows classically from (3.9) which provides the finiteness of  $\Gamma_\ell(\lambda, B)$  since  $m_\delta(B) < +\infty$  once  $\mathcal{L}^d(B) < +\infty$ . It remains to prove the upper bound in (3.9) which clearly follows from (2.4). Indeed, in term of  $\zeta = \frac{\#(S)}{m_\delta(B)}$ , we have for every finite  $S \subset B$ :

$$\lambda \#(S) - \xi_\ell(S) \leq m_\delta(B) (\lambda \zeta - \ell_-(\delta) \zeta (\zeta - 1)_+).$$

Taking the supremum of the right hand member with respect to  $\zeta \geq 0$  and noticing that  $\zeta (\zeta - 1)_+ = \varphi^*(\zeta)$ , we derive (3.9) after some easy manipulations. □

**Lemma 3.9.** *For every  $\lambda \geq 0$ , the set function  $B \mapsto \Gamma_\ell(\lambda, B)$  is sub-additive on disjoint Borel subsets and translation invariant.*

*Proof.* Let  $B_1, B_2$  be such that  $B_1 \cap B_2 = \emptyset$  and let  $S$  be a finite subset of  $B_1 \cup B_2$ . Then, setting  $S_i = S \cap B_i$ , we get a partition  $S = S_1 \cup S_2$  and by applying the super-additivity part of (2.1), we infer that:

$$\lambda \#(S) - \xi_\ell(S) \leq \sum_{i=1}^2 (\lambda \#(S_i) - \xi_\ell(S_i)) \leq \Gamma_\ell(\lambda, B_1) + \Gamma_\ell(\lambda, B_2),$$

hence the desired sub-additivity property by taking the supremum with respect to  $S \subset B_1 \cup B_2$ . The invariance by translation is trivial. □

In virtue of Lemma 3.9, we may now apply to  $\Gamma(\lambda, \cdot)$  a classical result by Krengel [16], which ensures the existence, for every  $\lambda$ , of a limit for the ratio  $\frac{\Gamma(\lambda, Q_k)}{k^d}$  as  $k \rightarrow +\infty$  (*thermodynamical limit*). Let us define the function

$$(3.10) \quad g_\ell(\lambda) := \inf_{k \in \mathbb{N}} \frac{\Gamma(\lambda, Q_k)}{k^d}.$$

Then we have:

$$(3.11) \quad g_\ell(\lambda) = \lim_{k \rightarrow +\infty} \frac{\Gamma(\lambda, Q_k)}{k^d} = \lim_{\varepsilon \rightarrow 0} \varepsilon^d \Gamma(\lambda, Q_{1/\varepsilon}).$$

For a proof of (3.11), we refer for instance to [18]. Since the effective profile  $f_\ell$  in Theorem 3.3 will be identified through the relation  $f_\ell^* = g_\ell$  (see the last step of the proof in Section 5), we now establish some useful bounds for  $g_\ell$ . Recalling the definition of  $\Lambda_\ell$  in (2.13), we introduce the function defined on  $\mathbb{R}$  by:

$$H_\ell(t) := \begin{cases} t \Lambda_\ell(t^{-\frac{1}{d}}) & \text{if } t > 0, \\ +\infty & \text{if } t \leq 0. \end{cases}$$

From  $(H_1)(H_2)$ , it is straightforward that this function  $H_\ell$  is l.s.c. and finite on  $(0, r_0^{-d}]$  while  $H_\ell(0_+) = H'_\ell(0_+) = 0$ . Note that  $H_\ell$  is not convex in general, even it could be discontinuous, as happens for  $\ell$  which is the step function  $\ell(r) = \frac{1}{2} \mathbf{1}_{[0,1]}$ . Indeed, in this case and for  $d = 1$ , we get  $H_\ell(t) = t [t]$  (see Remark 2.8). However  $H_\ell$  is convex continuous on  $\mathbb{R}_+$  in many classical cases including that of hyper singular Riesz potentials  $\ell(r) = r^{-s}$  where  $H_\ell(t) = C t^{1+s/d}$ .

**Proposition 3.10.** *The function  $g_\ell(\lambda)$  is convex, continuous, non-negative, vanishes on  $\mathbb{R}_-$  and, for  $\lambda > 0$ , satisfies the inequalities:*

$$(3.12) \quad H_\ell^*(\lambda) \leq g_\ell(\lambda) \leq \delta^{-d} \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

being  $\varphi$  given by (3.8) and  $\delta > 0$  chosen such that  $\ell_-(\delta) > 0$ .<sup>2</sup>

*Proof.* In view of (3.11) and of the convexity of  $\Gamma(\cdot, Q_k)$ , the function  $g_\lambda$  is convex, non-negative and vanishes for  $\lambda \leq 0$  as a pointwise limit as  $k \rightarrow \infty$  of the sequence of functions  $k^{-d} \Gamma(\cdot, Q_k)$ .

To prove the right hand inequality in (3.12), it is enough to apply (3.9) with  $B = Q_k$  and, after dividing by  $k^d$ , pass to the limit taking into account that, in virtue of (2.3),  $m_\delta(Q_k) \sim (\frac{k}{\delta})^d$  as  $k \rightarrow \infty$ . Since the majorant is a convex, continuous function of  $\lambda$ , we infer that  $g_\ell$  is continuous on  $\mathbb{R}$  as well.

Eventually, let us apply, for every  $t > 0$  and  $k \in \mathbb{N}_*$ , the fundamental upper bound (2.15) to the finite subset  $S_{t,k} := t^{-\frac{1}{d}} Z^d \cap Q_k$ . Then, recalling (3.7), we have:

$$\Gamma_\ell(\lambda, Q_k) \geq \lambda \#(S_{t,k}) - \xi_\ell(S_{t,k}) \geq \#(S_{t,k}) \left( \lambda - \Lambda_\ell(t^{-\frac{1}{d}}) \right).$$

Since  $\#(S_{t,k}) \sim t k^d$  as  $k \rightarrow \infty$  and in virtue of (3.11), we are led to:

$$g_\ell(\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(\lambda, Q_k)}{k^d} \geq t(\lambda - \Lambda_\ell(t^{-\frac{1}{d}})) = \lambda t - H_\ell(t).$$

The left hand side inequality in (3.12) follows by taking the supremum with respect to  $t$ .  $\square$

<sup>2</sup>The second inequality becomes  $g_\ell(\lambda) \leq \delta^{-d} \lambda$  in the case where  $\ell_-(\delta) = +\infty$ .

**Corollary 3.11.** *The integrand  $f_\ell := g_\ell^*$  is convex, l.s.c. and satisfies  $f_\ell(0) = 0$  while  $f_\ell(t) = +\infty$  for  $t < 0$ . Moreover, for every  $t > 0$  and  $\delta > 0$ , we have the inequalities:*

$$(3.13) \quad \ell_-(\delta) t (t\delta^d - 1)_+ \leq f_\ell(t) \leq H_\ell^{**}(t).$$

Accordingly,  $f_\ell$  is finite on  $[0, r_0^{-d}]$ , monotone non-decreasing on  $[0, +\infty)$  and satisfies:

$$(3.14) \quad f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \geq \sup_{\delta > 0} \ell_-(\delta)\delta^d > 0$$

*Proof.* Passing carefully to Fenchel conjugates in inequalities (3.12), we are led to (3.13) from which the other statements follow directly. In particular, as  $f'_\ell(0_+) = 0$ , we infer that the convex function  $f_\ell$  is monotone non-decreasing on  $\mathbb{R}_+$ .  $\square$

**Remark 3.12** (*growth conditions*). The inequality in (3.14) confirms that  $f_\ell$  grows at least quadratically at infinity, as announced in the introduction (see Theorem 3.3). More specifically, we can highlight two subcases for a cost  $\ell$  satisfying  $(H1) - (H3)$ .

- a)  $k_\ell := \sup_{\delta > 0} \ell_-(\delta)\delta^d = +\infty$ . Then  $\liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} = +\infty$  and  $f_\ell$  has a super quadratic growth. Note that this conclusion is consistent with the case  $\ell(r) = r^{-s}$  for  $s > d$  (see the next subsection).
- b)  $\int_0^\infty \ell_+(t)t^{d-1} dt < +\infty$ . In this case  $k_\ell < +\infty$  and, thanks to (3.13) and to the estimate given in Lemma 2.7 (that we can apply to  $\ell_+$  with  $r_0 = 0$ ), we obtain the lower and upper bounds:

$$0 < k_\ell \leq \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq \limsup_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq C \int_0^\infty \ell_+(t)t^{d-1} dt.$$

It follows that, under the integrability condition  $\int_{\mathbb{R}^d} \ell_+(|x|) dx < +\infty$ ,  $f_\ell$  enjoys a quadratic growth from above and from below.

### 3.4. Examples.

3.4.1. *The hard spheres model.* The hard spheres potential is given by

$$\ell(r) = \begin{cases} +\infty & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}$$

The computation of  $g_\ell$  through (3.10) and (3.11) leads to a linear function on  $\mathbb{R}_+$  namely  $g_\ell(\lambda) = \gamma_d \lambda$ , where  $\gamma_d$  denotes the densest spheres packing volume fraction in  $\mathbb{R}^d$ . This famous universal constant can be defined as

$$(3.15) \quad \gamma_d := \inf_{k \in \mathbb{N}^*} \frac{S(Q_k)}{k^d} = \lim_{k \rightarrow \infty} \frac{S(Q_k)}{k^d},$$

where, for any Borel set  $A \subset \mathbb{R}^d$ ,  $S(A)$  denotes the maximal number of points in  $A$  with mutual distance larger or equal to 1. The mean-field energy density  $f_\ell$  given by Theorem 3.3 is therefore the indicator function of the interval  $[0, \gamma_d]$

$$f_\ell(t) = 0 \quad \text{if } t \leq \gamma_d \quad , \quad f_\ell(t) = +\infty \quad \text{otherwise .}$$

Furthermore, for every continuous external potential  $U \in \mathcal{C}(\overline{\Omega})$ , we recover from Corollary 3.4 the convergence:

$$\min_{S \in \mathcal{F}_\varepsilon(\overline{\Omega})} \left\{ \varepsilon^d \sum_{z \in S} U(z) \right\} \rightarrow \gamma_d \int_{\overline{\Omega}} U(z) dz ,$$

where  $\mathcal{F}_\varepsilon(\overline{\Omega})$  is the family of finite subsets  $S \subset \overline{\Omega}$  satisfying  $|x - y| \geq \varepsilon$  for all  $(x, y) \in S^2 \setminus \Delta$ .

**Remark 3.13.** A variant of the previous result was obtained recently in [3] in the case where the total number of particles  $N_\varepsilon$  is prescribed to satisfy  $N_\varepsilon \varepsilon^d \rightarrow \kappa$  as  $\varepsilon \rightarrow 0$  where  $\kappa > 0$  is a given real parameter. With our notations this condition amounts to restrict the  $\Gamma$ -limit  $F$  to measures  $\rho$  such that  $\int_{\Omega} u dx = \kappa$ . Since the domain of  $F$  consists of density measures  $\rho = u \mathcal{L}^d \llcorner \Omega$  such that  $u \leq \gamma_d$  a.e., the latter integral condition requires that  $\kappa \leq \gamma_d |\Omega|$  hence a congestion ratio  $\theta := \frac{\kappa}{\gamma_d |\Omega|}$  not larger than 1. In this case and if, following the classical empirical measure representation,  $u$  is normalized to be a probability density by setting  $\tilde{u} := \frac{u}{\int_{\Omega} u dx} = \frac{u}{\kappa}$ , we recover a mean-field energy vanishing for  $\tilde{u} \leq \frac{\gamma_d}{\kappa}$  and infinite otherwise, exactly as stated in [3, Thm 6.1]. Note that the duality technique used there could only handle cost functions  $\ell$  taking values in the discrete set  $\{0, +\infty\}$ .

3.4.2. *The case of Riesz potentials.* Short range potential of Riesz type corresponds to fixing  $s > d$  and taking

$$l(r) = r^{-s} \text{ on } \mathbb{R}_+^d .$$

In this case, it's easy to establish from the homogeneity of the cost  $\ell$  that the bivariate function  $\Gamma_\ell$  defined in (3.7) satisfies, for every  $t \geq 0$ , the following scaling law :

$$(3.16) \quad \Gamma_\ell(t\lambda, B) = t \Gamma_\ell(\lambda, t^{1/s} B) .$$

It follows from (3.16) that:

$$\frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \frac{t \Gamma_\ell(\lambda, t^{1/s} Q_k)}{k^d} = t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} .$$

Sending  $k \rightarrow \infty$  and applying (3.11) two times, we get:

$$g_\ell(t\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \lim_{k \rightarrow \infty} t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} = t^{1+d/s} g_\ell(\lambda) .$$

In virtue of the equality  $f_\ell = g_\ell^*$ , we deduce that:

$$(3.17) \quad f_\ell(t) = C(s, d) t^{1+s/d}$$

where  $C(s, d) = f_\ell(1)$  is a universal constant. We thus recover the  $\Gamma$ -convergence result proved in [12, 13].

**Remark 3.14.** If we chose  $\ell(r) = r^{-s}$  where  $s < d$ , then condition (H3) is violated, and the scaling defined in equation (1.8) that we used to define  $F_\varepsilon$  will result in an infinite  $\Gamma$ -limit. This means that  $F(\rho)$  will be equal to  $+\infty$  whenever  $\rho$  is not equal to zero, and  $F(0)$  will be equal to zero. We can observe this when we consider a system of  $N_\varepsilon$  particles in  $S_\varepsilon \subset \bar{\Omega}$  such that  $\rho_\varepsilon := \rho_{S_\varepsilon}^\varepsilon$  converges weakly to  $\rho$ , and  $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < +\infty$ . Assuming  $\rho \neq 0$ , then we have  $N_\varepsilon \sim \|\rho\| \varepsilon^{-d}$  as  $\varepsilon \rightarrow 0$ . Moreover, due to the power law property of  $\ell$ , we can write:

$$(3.18) \quad F_\varepsilon(\rho_\varepsilon) = \varepsilon^{s+d} \xi_\ell(S_\varepsilon) \sim \frac{\|\rho\|^2}{\varepsilon^{d-s}} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2}.$$

As  $\ell$  satisfies (1.3), the convergence result of the long range case holds (with  $h_N = N^{-2}$ , see [3], [20]). Therefore, based on (1.4), and given that the standard empirical measure linked to  $S_\varepsilon$  converges to  $\hat{\rho} = \frac{\rho}{\|\rho\|} \in \mathcal{P}(\bar{\Omega})$ , it follows that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2} \geq D_\ell(\hat{\rho}) > 0.$$

This contradicts (3.18) since  $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < \infty$ . Therefore  $\rho = 0$ .

**3.4.3. The case of finite costs.** Many examples of finite costs can be considered as, for instance,  $\ell$  being a step function with compact support. Owing to Corollary 3.11, the effective convex integrand  $f_\ell$  has a quadratic growth on  $\mathbb{R}_+$ . The simplest one is the penalized version of the hard spheres potential defined by:

$$(3.19) \quad \ell(r) = \begin{cases} \frac{M}{2} & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad (M \text{ positive parameter}).$$

Applying the lower bound (3.13) with  $\delta = 1$ , we deduce that  $f_\ell \geq h$  where

$$h(t) := \frac{M}{2} t(t-1)_+.$$

In turn this lower bound is optimal for integer values of  $t$  since, as proved below,  $f_\ell$  coincides on  $\mathbb{R}_+$  with the piecewise affine interpolation of  $h$  given by:

$$(3.20) \quad f_\ell(t) = h(k) + (t-k)(h(k+1) - h(k)) \quad \forall t \in [k, k+1], k \in \mathbb{N}.$$

*Proof.* Owing to (3.11), the Fenchel conjugate of  $f_\ell$  is given by

$$g_\ell(\lambda) = \lim_{K \rightarrow +\infty} \frac{1}{K} \sup_{S \subset [0, K]} \left\{ \lambda \#(S) - \frac{M}{2} \#(\{(x, y) \in S^2 \setminus \Delta_1\}) \right\},$$

where  $\Delta_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \geq 1\}$ . Let  $S \subset [0, K]$  be an optimal set which we split in  $K$  disjoint pieces namely  $S = \bigcup_{i=1}^K S_i$  where  $S_i = S \cap [i, i + 1)$ . Let us denote  $n_i$  the number of points in  $S_i$ . By pushing them to the center of the interval  $[i, i + 1)$ , we see that the number of pairs in  $S^2 \setminus \Delta_1$  decreases to  $\sum_{i=1}^K n_i(n_i - 1)$ . It follows that:

$$\begin{aligned} g_\ell(\lambda) &= \lim_{K \rightarrow \infty} \sup_{n_i \in \mathbb{N}} \frac{1}{K} \left\{ \lambda \sum_{i=1}^K n_i - \frac{M}{2} \sum_{i=1}^K n_i(n_i - 1) \right\} \\ &= \sup_{n \in \mathbb{N}} \left\{ \lambda n - \frac{M}{2} n(n - 1) \right\} = (h + \chi_{\mathbb{N}})^*(\lambda), \end{aligned}$$

where  $\chi_{\mathbb{N}}$  denotes the indicator function of the integers. Therefore  $f_\ell = (g_\ell)^*$  is nothing else but the convexification of  $h + \chi_{\mathbb{N}}$  given by the interpolation formula (3.20).  $\square$

As demonstrated above, optimal point configurations for a constant external potential are obtained by periodically grouping a suitable number of points. Therefore, optimal sets are essentially multisets (see Remark 3.7). It is probable that a similar phenomenon occurs in higher dimensions.

The situation will vary if we consider a non-monotonic step function, such as the following:

$$\ell(r) = 1 \text{ for } r \in [0, 1], \quad \ell(r) = 4 \text{ for } r \in (1, 2), \quad \ell(r) = 0 \text{ for } r \geq 2.$$

In this case, we anticipate that optimal configurations may be periodic, but associated with a non-uniform Voronoi tessellation, consisting of patterns of different sizes, as observed in the context of optimal location problem (see [5, Sec 3.4]).

#### 4. PROOF OF THE MAIN THEOREM

First, we check the properness property (2.16) to make sure that the upper  $\Gamma$ -limit of  $F_\varepsilon$  is not trivial. To do this, it is sufficient to apply Lemma 2.9 by choosing  $u_0 = a \mathbf{1}_\Omega$  for  $a \in [0, r_0^{-d}]$ . Next, by virtue of the equi-coercivity property of  $F_\varepsilon$  proved in Proposition 2.5 and of the Kuratowski compactness theorem (see Proposition 3.1 and the introductory comment of Subsection 3.2), we can find a sequence  $\varepsilon_k \rightarrow 0$  and a weak\* lower semicontinuous functional  $F : \mathcal{M}_+(\Omega) \rightarrow [0, +\infty]$  such that  $F_{\varepsilon_k} \xrightarrow{\Gamma} F$  as  $k \rightarrow \infty$ . Note that the limit  $F$  may a priori depend on the chosen sequence  $\varepsilon_k \rightarrow 0$ . Accordingly, we will complete the proof of Theorem 3.3 in two steps which are outlined below:

**Step 1:** we show that  $F$  is a local functional of the form

$$(4.1) \quad F(\rho) := \begin{cases} \int_\Omega j(x, u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

where  $j : \Omega \times \mathbb{R} \rightarrow [0, +\infty]$  is a suitable convex normal integrand such that  $j(\cdot, 0) = 0$  a.e. in  $\Omega$ . As a consequence  $F$  is convex, weak\* l.s.c. and coincides with its Fenchel biconjugate, i.e. :

$$F(\rho) = F^{**}(\rho) = \sup_{v \in \mathcal{C}(\overline{\Omega})} \left\{ \int v d\rho - F^*(v) \right\}.$$

**Step 2:** we identify the Fenchel conjugate  $F^*$  in terms of the convex function  $g_\ell$  defined in(3.10), namely:

$$F^*(v) = \int_{\Omega} g_\ell(v) dx \quad , \quad \text{for every } v \in \mathcal{C}(\overline{\Omega}).$$

It follows that the limit functional  $F$  *does not depend on the sequence*  $(\varepsilon_k)$ . Also, since (3.14) the convex function  $f_\ell = g_\ell^*$  has a superlinear growth at infinity, by applying a classical result on convex functionals on measures (see for instance [6]), we obtain the equalities  $F = F^{**} = F_\ell$  where:

$$F_\ell(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

This will conclude the proof of the  $\Gamma$ -convergence of the whole sequence  $(F_\varepsilon)$  as stated in Theorem 3.3.

*Proof of Step 1.* Let  $\rho \in \mathcal{M}_+(\overline{\Omega})$  be such that  $F(\rho) < +\infty$ . Then, there exists a recovering sequence  $\rho_k \xrightarrow{*} \rho$  such that  $\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = F(\rho) < +\infty$ . By the assertion ii) of Proposition 2.5, we infer that  $\rho$  is an absolutely continuous measure. Accordingly, there exists a functional  $J : L^1(\Omega) \rightarrow [0, +\infty]$  such that

$$F(\rho) := \begin{cases} J(u) & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

The following result will be crucial for deriving the integral representation and the convexity of  $J$ . Its delicate proof is postponed to the end of this section.

**Lemma 4.1.** *The functional  $J : L^1(\Omega) \rightarrow [0, +\infty]$  defined above satisfies the following:*

- (i)  $J$  is weakly lower semicontinuous and satisfies  $J(0) = 0$ ;
- (ii) The domain of  $J$  is a subset of  $L^1(\Omega; \mathbb{R}_+)$  and  $J(u \mathbf{1}_A) \leq J(u)$  holds for every  $u \in L^1(\Omega, \mathbb{R}_+)$  and every Borel subset  $A \subset \Omega$ ;
- (iii) It holds  $J(u + v) = J(u) + J(v)$  whenever  $uv = 0$  is satisfied almost everywhere in  $\Omega$ .

In view of the assertions (i) and (iii) of Lemma 4.1 and since the Lebesgue measure on  $\Omega$  is atomless, we may apply a classical integral representation (see for instance Hiai and Umegaki [14, 15] or the monograph [8]) according to which there exists a suitable *convex* normal



integrand  $j$  such that (4.1) holds. Moreover, as  $J \geq 0$  and  $J(0) = 0$ , we have  $j(\cdot, 0) = 0$  a.e. while, due to the assertion (ii), the integrand  $j$  satisfies  $j(x, t) = +\infty$  if  $t < 0$ .  $\square$

*Proof of Step 2.* From Step 1 and by a classical result on integral functionals (see for instance [6]), the Fenchel conjugate of  $F$  is given for every  $v \in \mathcal{C}(\overline{\Omega})$  by:

$$F^*(v) = \sup_{u \in L^1(\Omega)} \left\{ \int_{\Omega} v u \, dx - \int_{\Omega} j(x, u(x)) \, dx \right\} = \int_{\Omega} j^*(x, v(x)) \, dx.$$

Obviously we may extend this equality to all functions  $v \in L^\infty(\Omega)$ . Noticing that  $j^*(x, 0) = -\inf j(x, \cdot) = -j(x, 0) = 0$ , we observe that, for every  $\lambda \in \mathbb{R}$  and for every hypercube  $Q(x_0, a) \subset \Omega$ , we have

$$F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = \int_{Q(x_0, a)} j^*(x, \lambda) \, dx.$$

Next we claim that, for any such an hypercube  $Q(x_0, a) \subset \Omega$ , the following holds:

$$(4.2) \quad F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda)$$

Suppose that this claim is true. Then, by considering Lebesgue points of  $j^*(\cdot, \lambda)$  for  $\lambda$  in a dense countable subset  $D$  of  $\mathbb{R}$ , we can find a Lebesgue negligible subset  $N \subset \Omega$  such that  $j^*(x, \lambda) = g_\ell(\lambda)$  for all  $(x, \lambda) \in (\Omega \setminus N) \times D$ . Thanks to the continuity of  $g_\ell$  proved in Proposition 3.10 and to the convexity of  $j^*(x, \cdot)$ , the latter equality can be then extended to all  $(x, \lambda) \in (\Omega \setminus N) \times \mathbb{R}$ , so that we have  $F^*(v) = \int_{\Omega} g_\ell(v) \, dx$  for every  $v \in \mathcal{C}(\overline{\Omega})$ . Hence the conclusion of Step 2 is reached and the proof of Theorem 3.3 is complete provided we can confirm (4.2).

We now focus on the proof of the equality (4.2). For  $\lambda \leq 0$ , this equality is trivial since  $g_\ell(\lambda) = j^*(x, \lambda) = 0$ . Next we observe that, for every  $\lambda \geq 0$  and  $Q(x_0, a) \subset \Omega$ , we have:

$$\begin{aligned} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) &:= \sup \{ \lambda \rho(Q(x_0, a)) - F_{\varepsilon_k}(\rho) : \rho \in \mathcal{M}_+(\overline{\Omega}) \} \\ &= \varepsilon_k^d \sup_{S \subset \overline{\Omega}} \{ \lambda \#(S \cap Q(x_0, a)) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S \subset Q(x_0, a)} \{ \lambda \#(S) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S' \subset Q(x_0, \frac{a}{\varepsilon_k})} \{ \lambda \#(S') - \xi_\ell(S'/\varepsilon_k) \} \\ &= \varepsilon_k^d \Gamma_\ell(\lambda, Q(x_0, \frac{a}{\varepsilon_k})) \end{aligned}$$

where:

- to pass from the second to the third line, we substitute any competitor  $S \subset \overline{\Omega}$  with  $S \cap Q(x_0, a)$  which has larger energy;

- to pass from the third line to the two last lines, we set  $S = S'/\varepsilon_k$  for going back from the  $\varepsilon_k$ - scaled energy (1.7) to the ground interaction energy  $\xi_\ell$  and ultimately recover the set function  $\Gamma_\ell$  defined in (3.7).

Therefore, thanks to (3.11), we can pass to the limit  $k \rightarrow +\infty$  (the position of  $x_0$  is irrelevant) and obtain the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda).$$

So, proving (4.2) reduces to checking the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = F^*(\lambda \mathbf{1}_{Q(x_0, a)}),$$

that we rewrite in the equivalent form:

$$(4.3) \quad \inf_{\rho \in \mathcal{M}_+(\overline{\Omega})} \{F_{\varepsilon_k}(\rho) - \lambda \rho(Q(x_0, a))\} \rightarrow \inf_{\rho} \{F(\rho) - \lambda \rho(Q(x_0, a))\}.$$

The left hand side infimum in (4.3) being non-positive (easily seen by taking  $\rho = 0$  as a competitor), we may apply Proposition 2.5 with the choice  $U = -\lambda \mathbf{1}_{Q(x_0, a)}$ . Therefore any minimizing sequence  $(\rho_k)$  for the left hand side of (4.3) is bounded in  $\mathcal{M}_+(\overline{\Omega})$  hence weakly\* relatively compact. By the assertion iii) of Proposition 3.1, we will be able to conclude the convergence of infima in (4.3) if we can show that

$$(4.4) \quad F_{\varepsilon_k} + G \xrightarrow{\Gamma} F + G \quad \text{being} \quad G(\rho) := -\lambda \rho(Q(x_0, a)).$$

In virtue of Theorem 3.3, we already know that  $F_{\varepsilon_k} \xrightarrow{\Gamma} F$ . Then it is enough to invoke the stability property of the assertion iv) of Proposition 3.1. However the functional  $G$  given above is not weak\* continuous on  $\mathcal{M}(\overline{\Omega})$  and therefore, we need to verify the less stringent requirements set out in (3.2). The first one is satisfied since  $|G(\rho)| \leq \lambda \|\rho\|$ . For the second one, we observe that, if  $F(\rho) < +\infty$ , then  $\rho$  is of the form  $\rho = u \mathcal{L}^d \llcorner \Omega$  (see Proposition 2.5), hence  $\rho(\partial Q(x_0, a)) = 0$  and every sequence  $\rho_n \xrightarrow{*} \rho$  satisfies  $\rho_n(Q(x_0, a)) \rightarrow \rho(Q(x_0, a)) = \int_{Q(x_0, a)} u dx$ . This confirms the validity of (4.4), hence that of (4.3). As a result the equality (4.2) is proved and, as announced, this achieves Step 2 and the proof of Theorem 3.3.  $\square$

*Proof of Lemma 4.1.* (i) Let  $u_k \rightarrow u$  in  $L^1(\Omega)$ . Then  $\rho_k = u_k \mathcal{L}^d \llcorner \Omega$  and  $\rho = u \mathcal{L}^d \llcorner \Omega$  are such that  $\rho_k \xrightarrow{*} \rho$  in  $\mathcal{M}_+(\overline{\Omega})$ . Since  $F = \Gamma\text{-}\lim F_{\varepsilon_k}$  is weak\* lower semicontinuous, we infer that

$$\liminf_{k \rightarrow \infty} J(\overline{u}_k) = \liminf_{k \rightarrow \infty} F(\rho_k) \geq F(\rho) = J(u).$$

To show that  $J(0) = 0$ , we consider as  $S_k$  a singleton  $\{x_0\}$  so that the associated measure  $\rho_k = (\varepsilon_k)^d \delta_{x_0}$  satisfies  $\rho_k \xrightarrow{*} 0$  while  $F_{\varepsilon_k}(\rho_k) = 0$ .

(ii) By the definition of the  $\Gamma$ -limit,  $F(\rho) < +\infty$  implies that  $\rho$  is a weak\* limit of a sequence  $(\rho_k)$  in  $\mathcal{M}_+(\overline{\Omega})$ , hence of the form  $\rho = u \mathcal{L}^d \llcorner \Omega$

with  $u \geq 0$ . Let now  $u \in L^1(\Omega; \mathbb{R}_+)$  and  $A$  a Borel subset of  $\Omega$ . We show first that  $J(u 1_A) \leq J(u)$  if  $\mathcal{L}^d(\partial A) = 0$ . We may assume that  $J(u) < +\infty$ . Hence, there exists a family of subsets  $S_k \subset \Omega$  such that  $\rho_k = \mu_{S_k}^{\varepsilon_k}$  satisfies  $\rho_k \xrightarrow{*} \rho := u \mathcal{L}^d \llcorner \Omega$  and  $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$ . If we let  $S'_k = S_k \cap A$  and  $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$ , we have  $\rho'_k \xrightarrow{*} u 1_A \mathcal{L}^d \llcorner \Omega$ , since indeed the convergence  $\rho_k \xrightarrow{*} \rho$  is tight while  $\rho(\partial A) = 0$ . Therefore,  $F_{\varepsilon_k}(\rho'_k) \leq F_{\varepsilon_k}(\rho_k)$ . By passing to the limit, as  $k \rightarrow +\infty$ , we deduce that

$$J(u 1_A) \leq F(\rho 1_A) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = J(u).$$

To extend the inequality to any Borel subset  $A$  of  $\Omega$ , it is enough to consider an approximating sequence  $(A_n)$  such that

$$\mathcal{L}^d(\partial A_n) = 0 \quad , \quad \mathcal{L}^d(A_n \Delta A) \rightarrow 0,$$

and then pass to the limit in the inequality  $J(u) \geq J(u 1_{A_n})$  while letting  $n \rightarrow +\infty$ . Indeed, the conclusion will then follow from the lower semicontinuity of  $J$  with respect to the norm convergence in  $L^1(\Omega)$ . Now, to construct such a sequence  $(A_n)$ , we consider a compact subset  $K_n \subset A$  and an open subset  $\omega_n \supset A$  such that  $\mathcal{L}^d(\omega_n \setminus K_n) \leq \frac{1}{n}$ . For every  $n$ , we can choose a suitable  $r_n > 0$  such that the enlarged open set  $A_n = K_n + B(0, r_n)$  satisfies  $\mathcal{L}^d(\partial A_n) = 0$ <sup>3</sup> while  $A_n \subset \omega_n$ . Then clearly  $\mathcal{L}^d(A_n \Delta A) \leq \mathcal{L}^d(\omega_n \setminus K_n) \rightarrow 0$ .

Let us now prove now the assertion (iii). In a first step, we assume that  $\text{spt}(u) \cap \text{spt}(v) = \emptyset$  so that there exists open subsets  $A \supset \text{spt}(u)$  and  $B \supset \text{spt}(v)$  such that  $\text{dist}(A, B) := \eta > 0$ .

We begin by proving the inequality  $J(u+v) \geq J(u) + J(v)$ . Without loss of generality, we may assume that  $J(u+v) < +\infty$  (hence  $u$  and  $v$  are non-negative). Then there exists a sequence of sets  $S_k \subset \bar{\Omega}$  such that

$$\rho_k = \mu_{S_k}^{\varepsilon_k} \xrightarrow{*} (u+v) \mathcal{L}^d \llcorner \Omega \quad , \quad F_{\varepsilon_k}(\rho_k) \rightarrow J(u+v).$$

We write  $S_k = S'_k \cup S''_k$  where  $S'_k = S_k \cap A$  and  $S''_k = S_k \cap B$  are disjoint. Then, we have  $\rho_k = \rho'_k + \rho''_k$  where  $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$  and  $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$ . Clearly  $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$  while  $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$ . Therefore, by applying the  $\Gamma$ -lim inf inequality to  $\rho'_k$  and  $\rho''_k$  while taking into account the super-additivity property of  $F_{\varepsilon_k}$  (see (2.1)), we deduce that:

$$\begin{aligned} J(u+v) &= \lim_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\geq F(u \mathcal{L}^d \llcorner \Omega) + F(v \mathcal{L}^d \llcorner \Omega) = J(u) + J(v). \end{aligned}$$

To show the converse inequality  $J(u+v) \leq J(u) + J(v)$ , we assume without any loss of generality that  $J(u) < +\infty$  and  $J(v) < +\infty$ . Then

<sup>3</sup>Here we use the fact that the function  $\alpha_n(r) = \mathcal{L}^d(\{x \in \Omega : \text{dist}(x, K_n) > r\})$  is bounded monotone non increasing so that it is continuous except possibly on a finite or countable subset of  $\mathbb{R}_+$ .

we consider recovering sequences  $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$  and  $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$  such that  $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$ ,  $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$  and  $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$ ,  $F_{\varepsilon_k}(\rho''_k) \rightarrow J(v)$ . Up to dropping the elements of  $S'_k$  which are not in  $A$  and the elements of  $S''_k$  which are not in  $B$ , we may assume that  $\text{spt}(\rho'_k) \subset A$  and  $\text{spt}(\rho''_k) \subset B$ . Indeed, removing these points will not affect the weak\* convergence to  $u \mathcal{L}^d \llcorner \Omega$  and  $v \mathcal{L}^d \llcorner \Omega$  respectively while the total energy  $\xi_{\varepsilon_k}$  will not increase. Therefore, by exploiting the right hand inequality in (2.1) and since  $\rho_k = \rho'_k + \rho''_k$  converge weakly\* to  $\rho = (u + v) \mathcal{L}^d \llcorner \Omega$ , we are led to the following set of inequalities:

$$\begin{aligned} J(u + v) &\leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\quad + \limsup_{k \rightarrow \infty} \frac{2\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d} \|u\|_{L^1(\Omega)} \|v\|_{L^1(\Omega)} \\ &\leq J(u) + J(v) + C \limsup_{k \rightarrow \infty} \frac{\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d}, \end{aligned}$$

where, in the second line, we used the tight convergence of  $\rho'_k, \rho''_k$ . The conclusion follows by noticing that  $\ell_+$  coincides with  $\ell$  for large values where it is non-increasing. Thus the integrability condition (H3) implies that  $r^d \ell(r) \rightarrow 0$  as  $r \rightarrow +\infty$ . The desired sub-additivity inequality follows.

In a second step, we remove the strict separation condition on the supports of  $u$  and  $v$  by simply assuming that the upper-level sets  $A := \{u > 0\}$  and  $B := \{v > 0\}$  satisfy  $\mathcal{L}^d(A \cap B) = 0$  (which, for  $u, v$  non-negative, is equivalent to say that  $uv = 0$  a.e.). To that aim, possibly after substituting  $A, B$  with non-intersecting Borel representatives, we consider increasing sequences of compact subsets  $K'_n \subset A, K''_n \subset B$  such that:

$$\mathcal{L}^d(A \setminus K'_n) \rightarrow 0, \quad \mathcal{L}^d(B \setminus K''_n) \rightarrow 0, \quad K'_n \cap K''_n = \emptyset.$$

Then  $u_n := u \mathbf{1}_{K'_n}, v_n := v \mathbf{1}_{K''_n}$  satisfy  $J(u_n + v_n) = J(u_n) + J(v_n)$ . In virtue of assertion (ii), we infer that, for every  $n$ :

$$J(u_n + v_n) \leq J(u) + J(v) \quad \text{and} \quad J(u_n) + J(v_n) \leq J(u + v).$$

In virtue of the lower semicontinuity of  $J$ , since  $u_n \rightarrow u$  and  $v_n \rightarrow v$  in  $L^1(\Omega)$ , we deduce from above the inequalities  $J(u + v) \leq J(u) + J(v)$  and  $J(u) + J(v) \leq J(u + v)$ , hence the desired additivity property.  $\square$

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