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MEAN FIELD THEORY FOR A GENERAL CLASS OF SHORT-RANGE INTERACTION FUNCTIONALS

GUY BOUCHITTÉ AND RAJESH MAHADEVAN

ABSTRACT. In models of N interacting particles in \mathbb{R}^d as in Density Functional Theory or crowd motion, the repulsive cost is usually described by a two-point function $c_\varepsilon(x, y) = \ell\left(\frac{|x-y|}{\varepsilon}\right)$ where $\ell : \mathbb{R}_+ \rightarrow [0, \infty]$ is decreasing to zero at infinity and parameter $\varepsilon > 0$ scales the interaction distance. In this paper we identify the mean-field energy of such a model in the short-range regime $\varepsilon \ll 1$ under the sole assumption that $\exists r_0 > 0 : \int_{r_0}^\infty \ell(r)r^{d-1} dr < +\infty$. This extends recent results [12, 13, 17] obtained in the homogeneous case $\ell(r) = r^{-s}$ where $s > d$.

Keywords: empirical measures, non-local functionals, Γ -convergence, mean-field energy, sub-additivity

Mathematics Subject Classification: 49J45, 49K21, 49N15, 60B10, 70-10, 82B21

1. INTRODUCTION

We consider a repulsive interaction function on $(\mathbb{R}^d)^N$ of the kind

$$(1.1) \quad c_N^\varepsilon(x_1, \dots, x_N) = \sum_{i \neq j} \ell\left(\frac{|x_i - x_j|}{\varepsilon}\right).$$

where:

- N is the number of particles in \mathbb{R}^d ;
- $\varepsilon > 0$ scales the interaction distance between particles.
- the two-particle cost $\ell : [0, +\infty] \rightarrow [0 + \infty]$ satisfies:

(H1) ℓ is l.s.c. and $\ell(0) > 0$ ($\ell(0) = +\infty$ is allowed)

(H2) $\exists r_0 \geq 0$ such that ℓ is finite and non increasing on $[r_0, +\infty)$ and $\lim_{r \rightarrow \infty} \ell(r) = 0$.

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In the whole paper, we denote by $\bar{\Omega}$ the closure of a smooth domain $\Omega \subset \mathbb{R}^d$ where the N particles are located. We speak of a *confined system* when $\bar{\Omega}$ is compact (container). Given a continuous exterior potential $U : \bar{\Omega} \rightarrow \mathbb{R}$, we consider the finite dimensional problem:

$$(1.2) \quad \mathcal{E}_N^\varepsilon(\Omega, U) := \inf \left\{ h_N c_N^\varepsilon(x_1, \dots, x_N) + \frac{1}{N} \sum_{i=1}^N U(x_i) : x_i \in \bar{\Omega} \right\},$$

where h_N is a suitable chosen normalization factor. Since the seminal work of Choquet in 1958 [9] and the growing interest of the quantum and statistical mechanics community, a lot of work has been devoted to the limit behavior of $\mathcal{E}_N^\varepsilon(\Omega, U)$ as $N \rightarrow \infty$ (ε fixed) as well as the characterization of the weak cluster points of the empirical measures associated with N -point configurations of minimal energy. The cornerstone of the mean field theory consists in identifying a limit energy functional on measures whose minimizers are precisely these cluster points.

1.1. State of the art. The scaling factor h_N in (1.2) must be selected so that the limit of the infimum belongs to $(0, +\infty)$. In turn this issue relies heavily on the integrability properties of the function $g(x) = \ell(|x|)$. Let us report on two cases of major interest:

1.1.1. Long range interaction case. Here $\Omega = \mathbb{R}^d$ and we take $\varepsilon = 1$. Moreover in addition to (H1)(H2), we assume that $g \in L_{loc}^1(\mathbb{R}^d)$ i.e.:

$$(1.3) \quad \int_0^1 r^{d-1} \ell(r) dr < +\infty.$$

In that case, a relevant choice is $h_N = \frac{1}{N^2}$ meaning roughly that the interaction energy is averaged over all pairs of distinct points in $\{x_1, x_2, \dots, x_N\}$. The identification of the mean-field energy is well known in the case of Riesz potentials $\ell(r) = \frac{1}{r^s}$ for $0 < s < d$, in the Logarithmic case $\ell(r) = -\log(r)$ for $d = 2$ and more generally for ℓ of positive type i.e. such that the Fourier transform of $g(x) = \ell(|x|)$ is positive in \mathbb{R}^d (see for instance the monograph by S. Serfaty [20]). It is given by a non-local functional, the so called *Direct energy*:

$$(1.4) \quad D_\ell(\rho) := \iint \ell(|x - y|) \rho \otimes \rho(dx dy).$$

Accordingly the limit problem associated with (1.2) reads:

$$\mathcal{E}_\infty(\Omega, U) = \inf \left\{ D_\ell(\rho) + \int U d\rho : \rho \in \mathcal{P}(\mathbb{R}^d) \right\},$$

where the infimum is reached at a unique configuration provided U grows suitably at infinity. At this stage, a few comments are in order:

- in the case of a confining external potential U , there are several impressive works devoted to the next order asymptotics [20, 19, 21, 10] in the case of Riesz potentials $\ell(r) = r^{-s}$ for $d \geq 3$ and $d - 2 \leq s < d$ revealing an asymptotic behavior as $N \rightarrow \infty$ of the form:

$$(1.5) \quad \lim_{N \rightarrow \infty} N^{1-\frac{s}{d}} (\mathcal{E}_N(\Omega, U) - \mathcal{E}_\infty(\Omega, U)) = C(s, d) \int (\rho_U)^{1+\frac{s}{d}},$$

where ρ_U is the unique minimizer realizing $\mathcal{E}_\infty(\Omega, U)$.

- if U remains bounded at infinity (for instance a Coulomb potential vanishing at infinity), the existence of a minimizer ρ_U may fail due to a loss of mass at infinity along minimizing sequences. A relaxation procedure leads to consider minimizers in the class of sub-probabilities $\rho \in \mathcal{P}_-(\mathbb{R}^d)$ and involves the weak* lower semicontinuous convexification of the Direct energy D . If ℓ is of positive type, this relaxed energy coincides with the natural 2-homogeneous extension of D to $\mathcal{P}_-(\mathbb{R}^d)$ while almost nothing is known if ℓ is merely locally integrable. For further details and examples of relaxed minimizers, we refer to the recent paper [3].

1.1.2. *Short range interaction case.* Following an idea developed for the hard spheres model [3], we look now at ε as a small parameter tending to zero with a prescribed speed as $N \rightarrow \infty$. If one thinks to a container $\bar{\Omega}$ of unit volume and ε to be the average distance of a particle to the others, it is natural to consider an asymptotic analysis where the product $\varepsilon^d N$ remains constant or converges to a given intensity factor $\kappa \in (0, +\infty)$. In a crowd model, this factor κ is related to a congestion ratio (see the hard spheres model in Section 3.4 and Remark 3.13). This of course means that we need to assume that $\varepsilon \sim N^{-\frac{1}{d}}$. Accordingly, in order to obtain a precise scaling for h_N ensuring a non-trivial behavior of the infimum (1.2), it is crucial to make an additional integrability on ℓ at infinity namely

$$(H3) \quad \int_{r_0}^{+\infty} \ell(r) r^{d-1} dr < +\infty.$$

It turns out that, under (H3), the right scaling factor in (1.2) is $h_N = \frac{1}{N}$ in contrast with the long range case. This covers the case of hyper singular Riesz potentials $\ell(r) = r^{-s}$ with $s > d$. For such potentials the parameter ε can be dropped thanks to the homogeneity and the normalized interaction energy becomes $\frac{1}{N^{1+\frac{s}{d}}} c_N^1(x_1, \dots, x_N)$. Under the latter scaling, it was proved recently [12, 13] that the mean field energy is a local functional defined on absolutely continuous measures $\rho = u \mathcal{L}^d \llcorner \Omega$ by $F(\rho) = C(s, d) \int_\Omega u^{1+\frac{s}{d}} d\mathcal{L}^d$, being $C(s, d)$ a universal constant. However, extending this result to more general costs seems to be difficult in the framework developed in [12], except possibly if ℓ is very close to a power potential.

1.2. Our contribution. This paper proposes a significant simplification of asymptotic analysis in the short range case. The approach is based on two components: first, treating the *interaction distance* ε in (1.1) as an infinitesimal parameter, and second, using an ε -counterpart of the traditional empirical measure frequently utilized in mean-field theory.

Thus, for every cost ℓ satisfying (H3), we can determine the mean field energy in terms of a *local* integral functional of the type $\int_{\Omega} f_{\ell}(u) dx$, where $u = \frac{d\rho}{dx}$ denotes the local particle density and f_{ℓ} is a convex integrand that exhibits super-linear growth at infinity. This expands upon previous findings [3] that were limited to the hard-spheres model (where $\ell(r) = +\infty$ when $r < 1$ and $\ell(r) = 0$ otherwise). Similarly, this permits to handle the case of hyper-singular Riesz potentials $\ell(r) = r^{-s}$ for $s > d$ analyzed in [12, 13].

It is noteworthy that fulfilling the integrability condition (H3) is crucial and cannot be sidestepped. When assuming that Ω is bounded, a cost ℓ that satisfies $\int_{r_0}^{+\infty} r^{d-1} \ell(r) dr = +\infty$ would give rise to an infinite limit in (1.2) if the scaling is by $h_N = \frac{1}{N}$ and $\varepsilon^d N \sim 1$ (see Remark 3.14).

1.3. Setting of the asymptotic problem and notations. From now on, Ω will be a *bounded* domain of \mathbb{R}^d with Lipschitz boundary ($\partial\Omega$ needs to be \mathcal{L}^d -negligible) and we consider a cost function ℓ which satisfies the standing assumptions (H1), (H2) and (H3).

For purposes of presentation, we utilize the infinitesimal length ε as the main parameter while the number of particles $N = N_{\varepsilon}$ approaches infinity as ε tends towards zero, following the scale $N_{\varepsilon} \sim \kappa \varepsilon^{-d}$ where κ is a positive constant. Later, we will establish that assigning a value to κ is not required because a uniform bound on the N_{ε} -point interaction energy will automatically result in $\limsup_{\varepsilon \rightarrow 0} N_{\varepsilon} \varepsilon^d < +\infty$. We can now

incorporate measures in $\overline{\Omega}$ into a variational framework for addressing the mean field problem. For each finite subset $S \subset \overline{\Omega}$, we define its ε -scaled empirical measure as follows:

$$(1.6) \quad \rho_S^{\varepsilon} := \varepsilon^d \sum_{x \in S} \delta_x.$$

This measure belongs to the set of non-negative Borel measures on $\overline{\Omega}$, denoted by $\mathcal{M}_+(\overline{\Omega})$. Here, $\|\rho\|$ represents the total mass, which may be infinite, of any element $\rho \in \mathcal{M}_+(\overline{\Omega})$. Through this, we can observe that $\|\rho_S^{\varepsilon}\| = \varepsilon^d \#(S)$, which could deviate from the classical empirical measure of S_{ε} with a total mass equal to 1. A key advantage of this approach is that it allows to avoid the non-local constraint that all competitors must belong to the subclass $\mathcal{P}(\overline{\Omega})$ of probability measures.

Next, we define the ε -scaled interaction energy of a discrete set $S \subset \mathbb{R}^d$ as follows:

$$(1.7) \quad \xi_{\ell,\varepsilon}(S) = \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left(\frac{|x-y|}{\varepsilon} \right) \quad \text{where} \quad \Delta := \{(x,x) : x \in \mathbb{R}^d\}.$$

We will refer to the interaction energy corresponding to $\varepsilon = 1$ as the “ground interaction energy”, denoted by ξ_ℓ . When the cost function ℓ is fixed, we will use ξ_ε instead of $\xi_{\ell,\varepsilon}$. With this in mind, we can define a scaled energy functional $F_\varepsilon : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$ for every $\varepsilon > 0$ in the following way:

$$(1.8) \quad F_\varepsilon(\rho) = \begin{cases} \varepsilon^d \xi_\varepsilon(S) & \text{if } \exists S \subset \overline{\Omega} \text{ such that } \rho = \rho_S^\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

The discrete problem (1.2) for $h_N = \frac{1}{N}$ and $N \varepsilon^d \sim \kappa$ can then be expressed through the relation:

$$\kappa \mathcal{E}_N^\varepsilon(\Omega, U) \sim \inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho : \rho \in \mathcal{M}_+(\overline{\Omega}) \right\} \quad \text{as } \varepsilon \rightarrow 0.$$

Accordingly the mean-field energy will be represented by a functional $F : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$ characterized by the property that, for every $U \in \mathcal{C}(\overline{\Omega})$, one has the convergence of infima

$$\inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho \right\} \rightarrow \inf \left\{ F(\rho) + \int_{\overline{\Omega}} U d\rho \right\}$$

accompanied by the tight convergence of minimizers. This falls squarely within the Γ -convergence theory ([2, 11, 7]) on which we rely to support our results.

The paper is organized as follows: in Section 2, we establish a lower bound which allows to obtain the strong equi-coercivity of the sequence (F_ε) ; in addition we show that any weak* cluster point of a sequence (ρ_ε) with uniformly bounded energy is absolutely continuous with respect to the Lebesgue measure; in Section 3, we state the Γ -convergence of F_ε as $\varepsilon \rightarrow 0$ to a convex functional of the form $F(\rho) = \int_{\overline{\Omega}} f_\ell \left(\frac{d\rho}{dx} \right) dx$, where the effective integrand f_ℓ grows at least quadratically at infinity. It is given by the thermodynamical limit of a subadditive set function (*Krengel's theorem*). Some examples and applications are given. The Section 4 is devoted to the proof of the main theorem.

Notations:

- $B(x, r)$ is the open ball of the Euclidean space \mathbb{R}^d centered at x and of radius r ; if $x = 0$, we simply denote B_r ;
- Q_k denotes the hypercube $[-k/2, k/2]^d$, $Q(x_0, r) := x_0 + r Q_1$;
- $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$ stands for the diagonal of \mathbb{R}^d ;

- $\sharp(S)$ denotes the counting measure of a subset $S \subset \mathbb{R}^d$ ($+\infty$ if S is infinite);
- \mathcal{L}^d is the Lebesgue measure in \mathbb{R}^d ; given any Borel set B , $|B|$ is a short notation for $\mathcal{L}^d(B)$; ω_d is such that $|B(x, r)| = \omega_d r^d$;
- $\mathcal{C}(\overline{\Omega})$ denotes the Banach space of continuous functions on the compact subset $\overline{\Omega}$ equipped with the uniform norm;
- $\mathcal{M}(\overline{\Omega})$ stands for the space of signed Radon measures on $\overline{\Omega}$ equipped with the total variation norm;
- $\mathcal{P}_-(\overline{\Omega})$ (resp. $\mathcal{P}(\overline{\Omega})$) is the subset of Borel measures $\mu \in \mathcal{M}_+(\overline{\Omega})$ such that $\|\mu\| := \mu(\overline{\Omega}) \leq 1$ (resp. $\|\mu\| = 1$).
- The topological support of $\mu \in \mathcal{M}_+(\overline{\Omega})$ is denoted $\text{supp}(\mu)$ while $\mu \llcorner A$ represents its restriction to a Borel subset $A \subset \overline{\Omega}$;
- The bracket $\langle \cdot, \cdot \rangle$ will denote the duality between $\mathcal{C}(\overline{\Omega})$ and $\mathcal{M}(\overline{\Omega})$:

$$\langle v, \mu \rangle = \int v d\mu,$$

This duality induces the weak* topology on $\mathcal{M}(\overline{\Omega})$ which can be identified with the dual of $\mathcal{C}(\overline{\Omega})$; as $\overline{\Omega}$ is compact, the weak* convergence $\mu_h \xrightarrow{*} \mu$ in $\mathcal{M}_+(\overline{\Omega})$ implies the tight convergence since $\|\mu_h\| = \langle 1, \mu_h \rangle \rightarrow \langle 1, \mu \rangle = \|\mu\|$.

- To any non-empty set A , we associate the functions:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}, \quad \chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}.$$

2. ENERGY ESTIMATES AND COMPACTNESS.

We begin with some elementary properties of the set function $\xi_\ell(S)$ (defined by (1.7) for $\varepsilon = 1$).

Lemma 2.1. *Let S_1, S_2 be finite disjoint subsets of \mathbb{R}^d . Then we have:*

(i) (super-additivity)

$$\xi_\ell(S_1 \cup S_2) \geq \xi_\ell(S_1) + \xi_\ell(S_2).$$

(i) (sub-additivity at large distance)

$$\xi_\ell(S_1 \cup S_2) \leq \xi_\ell(S_1) + \xi_\ell(S_2) + 2\ell_+(\eta) \sharp(S_1) \sharp(S_2),$$

where $\eta := \text{dist}(S_1, S_2)$.

Proof. Since S_1 and S_2 are non-intersecting, we can split $(S_1 \cup S_2)^2$ in four disjoint pieces as follows:

$$(S_1 \cup S_2)^2 = (S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_1 \times S_2) \cup (S_2 \times S_1).$$

The inequality (i) is then straightforward whereas, for the (ii), we simply majorize by $\ell_+(\eta)$ the contribution $\ell(|x - y|)$ of each pair (x, y) in $(S_1 \times S_2) \cup (S_2 \times S_1)$ where $|x - y| \geq \eta$ holds. \square

Recalling the notations (1.6) and (1.8) given in the introduction, we may rewrite a ε - rescaled version of Lemma 2.1 as follows: for every $\varepsilon > 0$ and all pairs of non-intersecting subsets $(S'_\varepsilon, S''_\varepsilon)$, one has

$$(2.1) \quad F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon + \rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) + \frac{2\ell_+(\eta_\varepsilon)}{\varepsilon^d} \|\rho'_\varepsilon\| \|\rho''_\varepsilon\|,$$

where $\rho'_\varepsilon := \rho_{S'_\varepsilon}^\varepsilon$, $\rho''_\varepsilon := \rho_{S''_\varepsilon}^\varepsilon$ and $\eta_\varepsilon = \varepsilon^{-1} \text{dist}(S'_\varepsilon, S''_\varepsilon)$.

2.1. A fundamental lower-bound. Since we are only assuming that $\ell(r)$ is decreasing for suitably large r , we need to define :

$$(2.2) \quad \ell_+(r) := \sup \{ \ell(s) : s \geq r \} \quad , \quad \ell_-(r) := \inf_{x,y \in [0,r]^d} \ell(|x-y|)$$

which are monotone non-increasing and satisfy:

$$\ell_+(r) = \ell(r) \quad \forall r \geq r_0 \quad , \quad \ell_-(r) = \inf \left\{ \ell(s) : s \leq r\sqrt{d} \right\} \leq \ell_+(r\sqrt{d}).$$

Next, in the same line as in the survey [17], we derive a very simple but fundamental lower bound for $\xi_\ell(S)$ when S is a N -point system contained in a Borel subset B of finite volume in \mathbb{R}^d . For such a B and any $\delta > 0$, we denote by $m_\delta(B)$ the minimal number of disjoint δ -hypercubes $Q_j = x_j + [-\delta/2, \delta/2]^d$ such that $B \subset \cup_{j=1}^{m_\delta(B)} \overline{Q_j}$. Then it's easy to check that $m_\delta(B) \sim \delta^{-d} \mathcal{L}^d(B)$ as $\delta \rightarrow 0$. For such subsets B , we will often use the following equivalent version obtained by keeping δ fixed while using large dilations:

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^d m_\delta \left(\frac{B}{\varepsilon} \right) = \delta^{-d} \mathcal{L}^d(B).$$

Lemma 2.2. *Let $B \subset \mathbb{R}^d$ be a Borel subset such that $\mathcal{L}^d(B) < +\infty$. Then for any N -point system $S \subset B$ and any $\delta > 0$, we have*

$$(2.4) \quad \xi_\ell(S) \geq N \ell_-(\delta) (\zeta - 1)_+ \quad \text{where} \quad \zeta = \frac{N}{m_\delta(B)}.$$

Proof. Let $\{Q_j, 1 \leq j \leq m_\delta(B)\}$ be a covering of B by disjoint δ -hypercubes Q_j and denote $n_j = \#(S \cap Q_j)$. We have $N = \sum n_j$ while, by the super additivity of ξ_ℓ (see Lemma 2.1) and the definition of $\ell_-(\delta)$, we have:

$$\xi_\ell(S) \geq \sum_{j=1}^{m_\delta(B)} \xi_\ell(S \cap Q_j) \geq \ell_-(\delta) \sum_{j=1}^{m_\delta(B)} n_j(n_j - 1).$$

If $N > m_\delta(B)$ (i.e. $\zeta > 1$), the desired lower bound (2.4) follows by noticing that the infimum

$$\inf \left\{ \sum_{j=1}^{m_\delta(B)} t_j(t_j - 1) : \sum_{j=1}^{m_\delta(B)} t_j = N, t_j \in \mathbb{R} \right\}$$

is reached for $t_j = \frac{N}{m_\delta(B)} = \zeta, \forall j$.

If $\zeta \leq 1$ the inequality (2.4) is trivial since $\xi_\ell(S) \geq 0$. \square

Remark 2.3. By the lower semicontinuity assumption (H1), since $\ell_-(0) = \ell(0) \in (0, +\infty]$, we may always find a $\delta > 0$ such that $\ell_-(\delta) > 0$. Note that the inequality (2.4) is still valid if $\ell_-(\delta) = +\infty$ provided we agree that $0 \times +\infty = 0$ (this situation occurs when $\ell = +\infty$ on an interval $[0, \delta_0)$).

2.2. Strong coercivity and compactness. We recall the definition of the functional F_ε given in (1.8) whose domain consists of ε -empirical measures of discrete subsets $S_\varepsilon \subset \overline{\Omega}$ (see the definition (1.6)).

Lemma 2.4 (strong coercivity). *Assuming that ℓ satisfies (H1) (H2), let $\delta > 0$ be such that $\ell_-(\delta) \in (0, +\infty]$ and set $\alpha := \frac{\delta^d \ell_-(\delta)}{\mathcal{L}^d(\Omega)}$.*

Then we have:

$$(2.5) \quad \liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|^2} \geq \alpha,$$

whenever (ρ_ε) is a sequence such that $\|\rho_\varepsilon\| \rightarrow +\infty$.

Proof. Without loss of generality, we may assume that $F_\varepsilon(\rho_\varepsilon) < +\infty$. Thus $\rho_\varepsilon = \rho_{S_\varepsilon}^\varepsilon$ for a suitable N_ε -point system $S_\varepsilon \subset \overline{\Omega}$ while:

$$F_\varepsilon(\rho) = \varepsilon^d \xi_\varepsilon(S_\varepsilon) = \varepsilon^d \xi_\ell \left(\frac{S_\varepsilon}{\varepsilon} \right), \quad \|\rho_\varepsilon\| = N_\varepsilon \varepsilon^d.$$

By applying the lower bound (2.4) to the subset $\varepsilon^{-1}S \subset \varepsilon^{-1}\overline{\Omega}$, we get:

$$(2.6) \quad F_\varepsilon(\rho_\varepsilon) \geq \ell_-(\delta) \|\rho_\varepsilon\| \left(\frac{\|\rho_\varepsilon\|}{\beta_\varepsilon} - 1 \right)_+ \quad \text{where } \beta_\varepsilon = \varepsilon^d m_\delta \left(\frac{\overline{\Omega}}{\varepsilon} \right).$$

From (2.3), we know that $\beta_\varepsilon \rightarrow \delta^{-d} \mathcal{L}^d(\Omega)$. Hence (2.5) follows by dividing (2.6) by $\|\rho_\varepsilon\|^2$ and passing to the limit as $\varepsilon \rightarrow 0$. \square

Proposition 2.5 (compactness). *Assuming that ℓ satisfies (H1) and (H2), let $U : \overline{\Omega} \rightarrow \mathbb{R}$ be a bounded Borel function and (ρ_ε) a sequence in $\mathcal{M}_+(\overline{\Omega})$ such that*

$$(2.7) \quad \sup_\varepsilon \left(F_\varepsilon(\rho_\varepsilon) + \int U d\rho_\varepsilon \right) < +\infty.$$

Then :

(i) *there exists a constant C such that*

$$\|\rho_\varepsilon\| + F_\varepsilon(\rho_\varepsilon) \leq C < +\infty \quad \text{for every } \varepsilon > 0.$$

(ii) *any weak* cluster point of (ρ_ε) is of the form $\rho = u \mathcal{L}^d \llcorner \Omega$ with $u \in L^1(\Omega)$.*

Remark 2.6. The proposition above implies that sequences of point configurations S_ε with equi-bounded energies admit a finite limiting intensity factor $\kappa := \limsup_\varepsilon \varepsilon^d \sharp(S_\varepsilon)$ while, by the assertion (ii), S_ε is not allowed to concentrate anywhere as $\varepsilon \rightarrow 0$.

Proof. Suppose that $\|\rho_\varepsilon\|$ has no upper bound. Then the uniform energy upper bound (2.7) implies that

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|} \leq \sup_{\bar{\Omega}} |U|,$$

while by (2.5) the left hand member of the previous inequality is infinite. So there is a contradiction and we can conclude that (ρ_ε) is bounded. Then it follows from (2.7) that $F_\varepsilon(\rho_\varepsilon)$ is bounded as well, whence the assertion (i).

Let us now prove the assertion (ii); we know that the sequence (ρ_ε) is bounded and therefore admits weak* cluster points. Given such a cluster point, we can assume, without loss of generality, that $\rho_\varepsilon \xrightarrow{*} \rho$ in $\mathcal{M}_+(\bar{\Omega})$. Let us introduce for every $t > 0$ the set

$$E_t := \left\{ x \in \bar{\Omega} : \liminf_{r \rightarrow 0} \frac{\rho(B(x, r))}{\omega_d r^d} > t \right\}.$$

Thanks to the upper semi-continuity of the map $x \rightarrow \rho(B(x, r))$, we infer that E_t is a Borel subset of $\bar{\Omega}$. We are going to prove that

$$(2.8) \quad \lim_{t \rightarrow +\infty} \rho(E_t) = 0.$$

To that aim, we consider the family of closed balls in R^d defined by

$$\mathcal{F}_t := \bigcup \left\{ \overline{B(x, r)} : x \in E_t, r < r_x, \rho(\partial B(x, r)) = 0 \right\},$$

where $r_x > 0$ is chosen so that $\rho(B(x, r)) > t \omega_d r^d$ for every $r < r_x$. Since \mathcal{F}_t determines a fine covering of the bounded Borel set E_t , we may invoke the Vitali-Besicovitch covering theorem (see [1, Thm 2.19]) which provides the existence a countable subfamily (B_n) such that

$$(2.9) \quad \rho(B_n) > t \mathcal{L}^d(B_n) \quad \forall n \quad , \quad \rho(E_t \setminus \cup_n B_n) = 0.$$

Next we associate with the weak* convergent sequence (ρ_ε) , two set functions defined on Borel subsets $A \subset \bar{\Omega}$:

$$\eta_\varepsilon(A) := F_\varepsilon(\rho_\varepsilon \llcorner A) \quad , \quad \eta(A) := \liminf_{\varepsilon \rightarrow 0} \eta_\varepsilon(A).$$

We can readily check that η_ε and η are monotone with respect to the inclusion while $\eta_\varepsilon(\bar{\Omega}) = F_\varepsilon(\rho_\varepsilon)$ implies that $\eta(\bar{\Omega}) \leq \beta < +\infty$. Moreover, by the first inequality in (2.1), η_ε is super-additive on disjoint Borel subsets. Obviously this holds true also for the set function η . By

applying this property to the sequence of disjoint balls B_n , we get the upperbound:

$$(2.10) \quad \sum_n \eta(B_n) \leq C .$$

On the other hand, thanks to the coercivity inequality (2.6) that we apply with $\bar{\Omega} = B_n$, we obtain:

$$(2.11) \quad \eta_\varepsilon(B_n) = F_\varepsilon(\rho_\varepsilon \llcorner B_n) \geq \ell_-(\delta) \rho_\varepsilon(B_n) \left(\frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} - 1 \right)_+ ,$$

where $\beta_\varepsilon(B_n) := \varepsilon^d m_\delta \left(\frac{B_n}{\varepsilon} \right)$. Since $\rho(\partial B_n) = 0$ by construction and thanks to (2.3) and (2.9), we infer that

$$\lim_{\varepsilon \rightarrow 0} \frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} = \frac{\rho(B_n)}{\delta^{-d} \mathcal{L}^d(\mathcal{B}_n)} \geq t \delta^d .$$

Therefore, passing to the limit $\varepsilon \rightarrow 0$ in (2.11), we deduce that:

$$(2.12) \quad \eta(B_n) \geq \ell_-(\delta) \rho(B_n) (t \delta^d - 1)_+ .$$

All in all, after collecting the second equality of (2.9), (2.10) and (2.12), we are led to:

$$\rho(E_t) \leq \sum_n \rho(B_n) \leq \frac{C}{\ell_-(\delta)(t \delta^d - 1)_+} .$$

Our claim (2.8) follows by sending $t \rightarrow +\infty$. The absolute continuity property $\rho \ll \mathcal{L}^d$ stated in the assertion (ii) is a consequence of the *Besicovitch differentiation theorem* [1, theorem 2.22]), which states that the singular part ρ_s in the Lebesgue-Nikodym decomposition of ρ with respect to the Lebesgue measure coincides with $\rho \llcorner E_\infty$ being $E_\infty = \bigcap_{t>0} E_t$. In our case $\|\rho_s\| = \rho(E_\infty) = 0$ due to (2.8). \square

2.3. Upper-bound of energies. In the same way as in [17], we will be using an upper bound of $\xi_\ell(S)$ when S is arranged on a d -dimensional periodic Bravais lattice \mathcal{G} ¹. To such a lattice we associate the ℓ -*Epstein zeta function* defined for every $r > 0$ by:

$$(2.13) \quad \Lambda_{\ell, \mathcal{G}}(r) := \sum_{x \in \mathcal{G} \setminus \{0\}} \ell(r|x|) .$$

In the case where the cartesian lattice $\mathcal{G} = \mathbb{Z}^d$ is used, we will write simply $\Lambda_\ell(r)$. The finiteness of this function for large r , under the condition (H3), turns out to be crucial for deriving an uniform upper bound for the scaled energy F_ε given in (1.8).

¹i.e. of the form $\mathcal{G} = F \mathbb{Z}^d$ for some invertible matrix $F \in \mathbb{R}^{d \times d}$

Lemma 2.7. *Under (H1) – (H3), there exists $C_{\mathcal{G}} > 0$ such that*

$$(2.14) \quad \Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \left(\ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt \right) \quad \forall r \geq r_0 \max\{1, a_{\mathcal{G}}^{-1}\},$$

where $a_{\mathcal{G}} := \min\{|y| : y \in \mathcal{G} \setminus \{0\}\}$.

Proof. To simplify, we chose the lattice \mathcal{G} so that $a_{\mathcal{G}} = 1$. Up to substituting ℓ with ℓ_+ which satisfies $\ell_+ \geq \ell$ and $\ell_+ = \ell$ on $[r_0, +\infty)$, we may also assume that ℓ is non-increasing on \mathbb{R}_+ . Accordingly, for any $s \in [0, \ell(0_+))$, the set of values $\{\ell > s\}$ forms a non-empty interval $[0, \ell^{-1}(s))$. The pseudo inverse $\ell^{-1}(s)$ is the supremum of all $t \geq 0$ such that $\ell(t) > s$, and it is a monotone non-increasing function on $[0, +\infty)$. Therefore, we have the following equivalence:

$$\ell^{-1}(s) > t \iff \ell(t) > s.$$

By applying the layer cake formula to the counting measure on \mathcal{G} , we get

$$\Lambda_{\ell, \mathcal{G}}(r) = \int_0^{\infty} N_r(s) ds,$$

where the integer function $N_r(s) := \#\{\{x \in \mathcal{G} \setminus \{0\} : \ell(r|x|) > s\}$ satisfies $N_r(s) = 0$ if $r \geq r_0$ and $s \geq \ell(r_0)$ (we assumed that $a_{\mathcal{G}} = 1$). On the other hand, for any periodic Bravais lattice $\mathcal{G} \subset \mathbb{R}^d$, there exists a constant $C_{\mathcal{G}} > 0$ such that

$$\#\{B_r \cap \mathcal{G}\} \leq C_{\mathcal{G}} r^d, \quad \forall r > 0.$$

This implies the inequality:

$$N_r(s) = \#\{\{x \in \mathcal{G} \setminus \{0\} : r|x| < \ell^{-1}(s)\}\} \leq \frac{C_{\mathcal{G}}}{r^d} (\ell^{-1}(s))^d.$$

Therefore, for every $r \geq r_0$, we are led to:

$$\Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds.$$

Then, after noticing that $\{s \in [0, \ell(r_0)] : \ell^{-1}(s) > t\} = [0, \ell(r_0) \wedge \ell(t)]$ holds for any $t \geq 0$, we obtain the desired inequality by applying once again the layer cake formula:

$$\begin{aligned} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds &= d \int_0^{\infty} (\ell(r_0) \wedge \ell(t)) t^{d-1} dt \\ &= \ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt. \end{aligned}$$

□

Remark 2.8. In view of Lemma 2.7, the ℓ - Epstein zeta function $\Lambda_{\ell, \mathcal{G}}(r)$ vanishes at infinity. However the behavior in $O(r^{-d})$ as $r \rightarrow \infty$ suggested by (2.14) is not optimal as we can see in the case of a Riesz potential $\ell(r) = r^{-s}$ with $s > d$, where $\Lambda_{\ell, \mathcal{G}}(r) = C r^{-s}$. On the other

hand, it is noteworthy that $\Lambda_{\ell, \mathcal{G}}$ is not continuous in general. A very simple example to see this is given by the step function $\ell = \frac{1}{2} \mathbf{1}_{[0,1]}$ which satisfies (H1) – (H3). For $d = 1$ and the lattice $\mathcal{G} = \mathbb{Z}$, we find that

$$\Lambda_{\ell}(r) = \#\{n \in \mathbb{N} : 0 < nr < 1\} = [r^{-1}] ,$$

where $[\cdot]$ denotes the integer part.

Next, by applying Lemma 2.7 in the case of the Cartesian lattice $\mathcal{G} = \mathbb{Z}^d$, we derive a fundamental upper bound for the short-range interaction energy.

Lemma 2.9. *Let $r > 0$ and S be a finite subset of the lattice $r\mathbb{Z}^d$. Then*

$$(2.15) \quad \xi_{\ell}(S) \leq \#(S) \Lambda_{\ell}(r)$$

As a consequence, for every $a > 0$, there exists $S_{\varepsilon} \subset \bar{\Omega}$ such that $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$ satisfies

$$(2.16) \quad \rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega \quad \text{and} \quad \limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|$$

where the right hand side upper bound is finite whenever $0 \leq a \leq r_0^{-d}$.

Proof. Let $S = \{r x_i : 1 \leq i \leq N\}$ where $N = \#(S)$ and $x_i \in \mathbb{Z}^d$. Noticing that, for every i , the set $\{x_i - x_j : j \neq i\}$ consists of $N - 1$ distinct elements of $\mathbb{Z}^d \setminus \{0\}$, we infer that

$$\sum_{j \neq i} \ell(r|x_i - x_j|) \leq \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \ell(r|z|) = \Lambda_{\ell}(r),$$

hence the desired inequality (2.15) by summing with respect to i .

Taking now $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$ where $S_{\varepsilon} = \Omega \cap (r_{\varepsilon}\mathbb{Z})^d$ and $r_{\varepsilon} = \varepsilon a^{-\frac{1}{d}}$, we obtain a sequence such that $\rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega$ as $\varepsilon \rightarrow 0$. Indeed, by the periodicity of the Euclidean lattice, ρ_{ε} converges to a uniform density on Ω while its total mass $\|\rho_{\varepsilon}\| = \varepsilon^d \#(S_{\varepsilon}) \sim \varepsilon^d (r_{\varepsilon}^{-d} |\Omega|)$ converges to $a |\Omega|$ as $\varepsilon \rightarrow 0$. Eventually, by applying (2.15), we get $F_{\varepsilon}(\rho_{\varepsilon}) = \varepsilon^d \xi_{\ell}(\frac{S_{\varepsilon}}{\varepsilon}) \leq \varepsilon^d N_{\varepsilon} \Lambda_{\ell}(a^{-1/d})$, whence:

$$\limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|.$$

The finiteness of $\Lambda_{\ell}(a^{-\frac{1}{d}})$ for $a \in [0, r_0^{-d}]$ follows from Lemma 2.7. \square

Remark 2.10. The upper bound (2.15) and (2.16) obtained by choosing $\mathcal{G} = \mathbb{Z}^d$ as the reference lattice are in general not optimal since the Epstein-zeta function of another lattice could provide better ones. Obviously the inequality (2.15) holds true after replacing Λ_{ℓ} with the Epstein function $\Lambda_{\ell, \mathcal{G}}$ of any Bravais lattice $\mathcal{G} = F\mathbb{Z}^d$, while for the validity of (2.15), we need to add the normalization condition $|\det(F)| = 1$

(thereby fixing the volume of the so called *fundamental domain* of \mathcal{G}). The existence and the determination of an optimal lattice \mathcal{G} for the following minimization problem:

$$\inf\{\Lambda_{\ell,\mathcal{G}}(r) : \mathcal{G} = F Z^d, |\det(F)| = 1\}$$

touches on a very hard and famous problem related to crystallisation conjectures (see [17, 4]). Note that, for a general cost ℓ , the answer to this problem will depend of the value r (thus of the local density $a = \frac{d\rho}{dx}$ of the limiting measure ρ).

3. THE Γ -CONVERGENCE RESULT.

3.1. A quick overlook. The notion of Γ -convergence is popular in the community of calculus of variations and very much used in the analysis of sharp-interface models, dimension reduction for problems in mechanics, optimal design and homogenization. As pointed out in the introduction, this tool is also perfectly suited to justify a mean-field approach for large particle systems subject to a minimum energy criterion (see [21]). For the convenience of the reader, let us give here some basic definitions and main properties. For further details, we refer to the monographs [2, 11, 7].

Let (E, τ) be a metrizable topological space and consider a sequence of functionals $F_n : E \rightarrow (-\infty, +\infty]$. Then the lower Γ -limit F_- and the upper Γ -limit F_+ of F_n are defined by:

$$F_-(u) := \inf_{u_n \rightarrow u} \liminf_{n \rightarrow \infty} F_n(u_n) \quad , \quad F_+(u) := \inf_{u_n \rightarrow u} \limsup_{n \rightarrow \infty} F_n(u_n).$$

Both are τ -lower semicontinuous (see [2]), whereas in general it holds $F_- \leq F_+$. In practice it is useful to check that these functionals are proper i.e. that they range into $\mathbb{R} \cup \{+\infty\}$ being not identically $+\infty$. If F_n admits a lower bound independent of n , this amounts to checking that the existence of $u_0 \in E$ such that

$$(3.1) \quad F_+(u_0) = \inf_{u_n \rightarrow u_0} \limsup_{n \rightarrow \infty} F_n(u_n) < +\infty.$$

We say that F_n Γ -converges to F (denoted $F_n \xrightarrow{\Gamma} F$) if $F = F_- = F_+$ or equivalently if the two following conditions are fulfilled:

- a) (*lowerbound*) For any sequence u_n converging to u , we have the inequality

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F(u);$$

- b) (*recovering sequence*) For every $u \in X$ such that $F(u) < +\infty$, there exists (u_n) such that

$$u_n \rightarrow u \quad \text{and} \quad F_n(u_n) \rightarrow F(u).$$

This is the case in particular if $F_n = F$ does not depend of n ; then $F_n \xrightarrow{\Gamma} \text{cl}(F)$ where $\text{cl}(F)$ denotes the τ -lower semicontinuous envelope of F . Among all properties of Γ -convergence, we give some which will be used in this paper.

Proposition 3.1. *Let $F_n : E \rightarrow (-\infty, +\infty]$ and assume (3.1). Then:*

- (i) $F_n \xrightarrow{\Gamma} F \iff \text{cl}(F_n) \xrightarrow{\Gamma} F$;
- (ii) (Kuratowski compactness Theorem) *If (E, τ) is a second countable topological space (for instance a separable metric space), then any sequence (F_n) admits a Γ -convergent subsequence;*
- (iii) (convergence of infima) *Suppose that $F_n \xrightarrow{\Gamma} F$ and that the following equi-coercivity property holds:*

$$\sup_n F_n(u_n) < +\infty \implies \{u_n\} \text{ is } \tau\text{-relatively compact.}$$

Then $\lim_{n \rightarrow \infty} \inf_X F_n = \min_X F$ and the minimum set for F coincides with the cluster points of all sequences (u_n) such that $F_n(u_n) - \inf F_n \rightarrow 0$;

- (iv) (stability) $F_n \xrightarrow{\Gamma} F \implies F_n + G \xrightarrow{\Gamma} F + G$ for every continuous perturbation function $G : E \rightarrow \mathbb{R}$.

Remark 3.2. The continuity requirement for G in the assertion (iv) is often too restrictive. Actually the same conclusion holds under the following milder condition:

$$(3.2) \quad \begin{cases} \inf_K G > -\infty & \text{for any compact } K \subset E \\ G(u_n) \rightarrow G(u) & \text{whenever } u_n \rightarrow u \text{ and } F(u) < +\infty \end{cases}$$

For the convenience of the reader, a brief proof of the sufficiency of this condition is given below.

Proof. To check condition a), we consider a sequence (u_n) such that $u_n \rightarrow u$. Without loss of generality, assume that $F_n(u_n) + G(u_n) \leq C$ for a suitable constant C . Since $G(u_n)$ is lower bounded (take K to be $\{u_n, n \in \mathbb{N}\} \cup \{u\}$), we infer that $F_n(u_n) \leq C'$ for another constant C' . By the Γ -convergence $F_n \xrightarrow{\Gamma} F$, it follows that

$$F(u) \leq \liminf_n F_n(u_n) < +\infty.$$

Therefore $G(u_n) \rightarrow G(u)$ and $\liminf_n (F_n + G)(u_n) \geq (F + G)(u)$.

For checking condition b), we may restrict to elements $u \in E$ such that $(F + G)(u) < +\infty$. Then $F(u) < +\infty$ and any recovering sequence $u_n \rightarrow u$ such that $F_n(u_n) \rightarrow F(u)$ will satisfy $G(u_n) \rightarrow G(u)$. \square

3.2. The main result. In our context, the Γ -convergence issue applies to the sequence (F_ε) defined in (1.8) and to the ambient topological space $\mathcal{M}_+(\bar{\Omega})$ embedded with the weak* topology (tight convergence). Thanks to the equi-coercivity property established in Proposition 2.5, there is no loss of generality in working in a fixed closed ball of $\mathcal{M}_+(\bar{\Omega})$ which is metrizable and compact. Therefore all the properties mentioned in the former subsection are applicable (after substituting the index $n \rightarrow \infty$ with the continuous parameter $\varepsilon \rightarrow 0$).

Theorem 3.3. *Let ℓ satisfy the standing assumptions (H1) – (H3) and let $F_\varepsilon : \mathcal{M}_+(\bar{\Omega}) \mapsto [0, +\infty]$ be given by (1.8). Then $F_\varepsilon \xrightarrow{\Gamma} F$ (for the weak* topology) where*

$$F(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

and $f_\ell : \mathbb{R}_+ \rightarrow [0, +\infty]$ is convex, l.s.c. and satisfies

$$(3.3) \quad f_\ell(0) = f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} > 0.$$

The proof of Theorem 3.3 is postponed to Section 5 while the property (3.3) is established in the next Sub-section. Let us now consider the Fenchel conjugate of f_ℓ (implicitly extended by $+\infty$ on $(-\infty, 0)$) given by

$$(3.4) \quad f_\ell^*(\lambda) := \sup \{ \lambda t - f_\ell(t) : t \in \mathbb{R}_+ \}$$

From (3.3), one can check that the supremum in (3.4) is actually a maximum (which is attained at $t = 0$ if $\lambda \leq 0$). Therefore, f_ℓ^* is convex, continuous and vanishes on $(-\infty, 0]$. As a consequence it admits left and right derivatives $(f_\ell^*)'(\lambda_-) \leq (f_\ell^*)'(\lambda_+)$ for any λ so that the subdifferential $\partial f_\ell^*(\lambda) = [(f_\ell^*)'(\lambda_-), (f_\ell^*)'(\lambda_+)]$ is non empty.

Corollary 3.4. *Let $U \in \mathcal{C}(\bar{\Omega})$ be an external potential. Then, for every $\varepsilon > 0$, there exists a finite set $S_\varepsilon \subset \bar{\Omega}$ minimizing*

$$\mathcal{I}_\ell^{(\varepsilon)}(U) := \inf_{S \subset \bar{\Omega}} \left\{ \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left(\frac{|x-y|}{\varepsilon} \right) + \sum_{x \in S} U(x) \right\}.$$

Moreover $\sup_\varepsilon \varepsilon^d \sharp(S_\varepsilon) < +\infty$ and $\lim_{\varepsilon \rightarrow 0} \varepsilon^d \mathcal{I}_\ell^{(\varepsilon)}(U) = \mathcal{I}_\ell(U)$, where

$$(3.5) \quad \mathcal{I}_\ell(U) := \min_{u \in L^1(\Omega)} \left\{ \int_{\Omega} (f_\ell(u) + uU) dx \right\} = - \int_{\Omega} f_\ell^*(-U) dx.$$

Furthermore any weak* cluster point of $\rho_{S_\varepsilon}^\varepsilon$ belongs to the minimum set of (3.5) given by

$$(3.6) \quad \mathcal{S}_\ell := \{ u \mathcal{L}^d \llcorner \Omega : u(x) \in \partial f_\ell^*(-U(x)) \text{ a.e. } x \in \Omega \}.$$

Remark 3.5 (*Non-uniqueness*). In general f_ℓ is not strictly convex and ∂f_ℓ^* can be multi-valued (see for instance the hard spheres case depicted in Subsection 3.4 or the example of the step function ℓ given in (3.19) where f_ℓ is piecewise affine). Note that, since f_ℓ^* vanishes on \mathbb{R}_- , it holds $\mathcal{I}_\ell(U) = 0$ for every non-negative potential U .

Remark 3.6. A natural variant of $\mathcal{I}_\ell^{(\varepsilon)}(U)$ consists in prescribing the total number N_ε of particles to satisfy $N_\varepsilon \sim \kappa \varepsilon^{-d}$ for some given intensity factor $\kappa \in (0, +\infty)$. Accordingly, Corollary 3.4 can be restated by adding a total mass constraint in the limit problem, that is with $I_\ell(U)$ in (3.5) replaced by

$$\mathcal{I}_{\ell,\kappa}(U) := \inf \left\{ \int_{\Omega} (f_\ell(u) + uU) dx : u \in L^1(\Omega; \mathbb{R}_+) , \int_{\Omega} u dx = \kappa \right\}.$$

The associated minimum $\mathcal{S}_{\ell,\kappa}$ can be determined by selecting a suitable Lagrange multiplier depending implicitly on κ ; as a consequence, an explicit form for $\mathcal{S}_{\ell,\kappa}$ of the kind (3.6) is not available.

Remark 3.7 (*Clustering*). For $\varepsilon > 0$ fixed, the lower semicontinuity property of F_ε requires that $\ell(0+) = +\infty$. Otherwise, if $\ell(0+) < +\infty$, a sequence of subsets $S_n \subset \bar{\Omega}$ such that $\sup_n F_\varepsilon(\rho_{S_n}^\varepsilon) < +\infty$ (thus retaining a finite number of points) can collapse into several clusters while retaining finite energy. In this case, the relaxed functional $\text{cl}(F_\varepsilon)$ can be obtained directly by extending its domain to ε -empirical measures associated with multisets (instead of sets) and by extending the definition (1.6) accordingly taking into account the multiplicity of each cluster of particles. Note however that considering $\text{cl}(F_\varepsilon)$ instead of F_ε will not change the mean-field energy F given in Theorem 3.3 in virtue of the assertion i) of Proposition 3.1.

3.3. Characterization and properties of f_ℓ . The convex integrand f_ℓ will be characterized indirectly through its Fenchel conjugate. For every $\lambda \in \mathbb{R}$ and any Borel subset $B \subset \mathbb{R}^d$, we define:

$$(3.7) \quad \Gamma_\ell(\lambda, B) := \sup \{ \lambda \#(S) - \xi_\ell(S) : S \text{ finite } \subset B \}.$$

The key properties of this bivariate function are summarized in the two following lemmas. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the convex, continuous function defined by:

$$(3.8) \quad \varphi(t) = \begin{cases} \frac{(1+t)^2}{4} & \text{if } t \geq 1 \\ t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t < 0 \end{cases}.$$

A straightforward computation shows that its Fenchel conjugate is given by:

$$\varphi^*(\zeta) = \zeta(\zeta - 1)_+ \quad \text{if } \zeta \geq 0 \quad , \quad \varphi^*(\zeta) = +\infty \quad \text{if } \zeta < 0.$$

Lemma 3.8. *Assume that $\mathcal{L}^d(B) < +\infty$. Then the map $\lambda \rightarrow \Gamma_\ell(\lambda, B)$ is convex, continuous, vanishes on \mathbb{R}_- and satisfies:*

$$(3.9) \quad 0 \leq \Gamma_\ell(\lambda, B) \leq m_\delta(B) \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

for every $\lambda \in \mathbb{R}$, being $\delta > 0$ such that $\ell_-(\delta) \in (0, +\infty]$ and φ being defined by (3.8).

Notice that the right hand side of (3.9) is convex as a function λ . On the other hand, due to the linear behavior of φ on $[0, 1]$, we can easily infer that $\Gamma_\ell(\lambda, B) \leq \lambda m_\delta(B)$ whenever $\ell_-(\delta) = +\infty$.

Proof. As a supremum of affine functions, $\Gamma_\ell(\lambda, B)$ is convex and l.s.c. with respect to λ ; it is non-negative (follows by taking S the empty set) and vanishes if $\lambda \leq 0$. The continuity property follows classically from (3.9) which provides the finiteness of $\Gamma_\ell(\lambda, B)$ since $m_\delta(B) < +\infty$ once $\mathcal{L}^d(B) < +\infty$. It remains to prove the upper bound in (3.9) which clearly follows from (2.4). Indeed, in term of $\zeta = \frac{\#(S)}{m_\delta(B)}$, we have for every finite $S \subset B$:

$$\lambda \#(S) - \xi_\ell(S) \leq m_\delta(B) (\lambda \zeta - \ell_-(\delta) \zeta (\zeta - 1)_+).$$

Taking the supremum of the right hand member with respect to $\zeta \geq 0$ and noticing that $\zeta (\zeta - 1)_+ = \varphi^*(\zeta)$, we derive (3.9) after some easy manipulations. □

Lemma 3.9. *For every $\lambda \geq 0$, the set function $B \mapsto \Gamma_\ell(\lambda, B)$ is sub-additive on disjoint Borel subsets and translation invariant.*

Proof. Let B_1, B_2 be such that $B_1 \cap B_2 = \emptyset$ and let S be a finite subset of $B_1 \cup B_2$. Then, setting $S_i = S \cap B_i$, we get a partition $S = S_1 \cup S_2$ and by applying the super-additivity part of (2.1), we infer that:

$$\lambda \#(S) - \xi_\ell(S) \leq \sum_{i=1}^2 (\lambda \#(S_i) - \xi_\ell(S_i)) \leq \Gamma_\ell(\lambda, B_1) + \Gamma_\ell(\lambda, B_2),$$

hence the desired sub-additivity property by taking the supremum with respect to $S \subset B_1 \cup B_2$. The invariance by translation is trivial. □

In virtue of Lemma 3.9, we may now apply to $\Gamma(\lambda, \cdot)$ a classical result by Krengel [16], which ensures the existence, for every λ , of a limit for the ratio $\frac{\Gamma(\lambda, Q_k)}{k^d}$ as $k \rightarrow +\infty$ (*thermodynamical limit*). Let us define the function

$$(3.10) \quad g_\ell(\lambda) := \inf_{k \in \mathbb{N}} \frac{\Gamma(\lambda, Q_k)}{k^d}.$$

Then we have:

$$(3.11) \quad g_\ell(\lambda) = \lim_{k \rightarrow +\infty} \frac{\Gamma(\lambda, Q_k)}{k^d} = \lim_{\varepsilon \rightarrow 0} \varepsilon^d \Gamma(\lambda, Q_{1/\varepsilon}).$$

For a proof of (3.11), we refer for instance to [18]. Since the effective profile f_ℓ in Theorem 3.3 will be identified through the relation $f_\ell^* = g_\ell$ (see the last step of the proof in Section 5), we now establish some useful bounds for g_ℓ . Recalling the definition of Λ_ℓ in (2.13), we introduce the function defined on \mathbb{R} by:

$$H_\ell(t) := \begin{cases} t \Lambda_\ell(t^{-\frac{1}{d}}) & \text{if } t > 0, \\ +\infty & \text{if } t \leq 0. \end{cases}$$

From $(H_1)(H_2)$, it is straightforward that this function H_ℓ is l.s.c. and finite on $(0, r_0^{-d}]$ while $H_\ell(0_+) = H'_\ell(0_+) = 0$. Note that H_ℓ is not convex in general, even it could be discontinuous, as happens for ℓ which is the step function $\ell(r) = \frac{1}{2} \mathbf{1}_{[0,1]}$. Indeed, in this case and for $d = 1$, we get $H_\ell(t) = t [t]$ (see Remark 2.8). However H_ℓ is convex continuous on \mathbb{R}_+ in many classical cases including that of hyper singular Riesz potentials $\ell(r) = r^{-s}$ where $H_\ell(t) = C t^{1+s/d}$.

Proposition 3.10. *The function $g_\ell(\lambda)$ is convex, continuous, non-negative, vanishes on \mathbb{R}_- and, for $\lambda > 0$, satisfies the inequalities:*

$$(3.12) \quad H_\ell^*(\lambda) \leq g_\ell(\lambda) \leq \delta^{-d} \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

being φ given by (3.8) and $\delta > 0$ chosen such that $\ell_-(\delta) > 0$.²

Proof. In view of (3.11) and of the convexity of $\Gamma(\cdot, Q_k)$, the function g_λ is convex, non-negative and vanishes for $\lambda \leq 0$ as a pointwise limit as $k \rightarrow \infty$ of the sequence of functions $k^{-d} \Gamma(\cdot, Q_k)$.

To prove the right hand inequality in (3.12), it is enough to apply (3.9) with $B = Q_k$ and, after dividing by k^d , pass to the limit taking into account that, in virtue of (2.3), $m_\delta(Q_k) \sim (\frac{k}{\delta})^d$ as $k \rightarrow \infty$. Since the majorant is a convex, continuous function of λ , we infer that g_ℓ is continuous on \mathbb{R} as well.

Eventually, let us apply, for every $t > 0$ and $k \in \mathbb{N}_*$, the fundamental upper bound (2.15) to the finite subset $S_{t,k} := t^{-\frac{1}{d}} Z^d \cap Q_k$. Then, recalling (3.7), we have:

$$\Gamma_\ell(\lambda, Q_k) \geq \lambda \#(S_{t,k}) - \xi_\ell(S_{t,k}) \geq \#(S_{t,k}) \left(\lambda - \Lambda_\ell(t^{-\frac{1}{d}}) \right).$$

Since $\#(S_{t,k}) \sim t k^d$ as $k \rightarrow \infty$ and in virtue of (3.11), we are led to:

$$g_\ell(\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(\lambda, Q_k)}{k^d} \geq t(\lambda - \Lambda_\ell(t^{-\frac{1}{d}})) = \lambda t - H_\ell(t).$$

The left hand side inequality in (3.12) follows by taking the supremum with respect to t . \square

²The second inequality becomes $g_\ell(\lambda) \leq \delta^{-d} \lambda$ in the case where $\ell_-(\delta) = +\infty$.

Corollary 3.11. *The integrand $f_\ell := g_\ell^*$ is convex, l.s.c. and satisfies $f_\ell(0) = 0$ while $f_\ell(t) = +\infty$ for $t < 0$. Moreover, for every $t > 0$ and $\delta > 0$, we have the inequalities:*

$$(3.13) \quad \ell_-(\delta) t (t\delta^d - 1)_+ \leq f_\ell(t) \leq H_\ell^{**}(t).$$

Accordingly, f_ℓ is finite on $[0, r_0^{-d}]$, monotone non-decreasing on $[0, +\infty)$ and satisfies:

$$(3.14) \quad f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \geq \sup_{\delta > 0} \ell_-(\delta)\delta^d > 0$$

Proof. Passing carefully to Fenchel conjugates in inequalities (3.12), we are led to (3.13) from which the other statements follow directly. In particular, as $f'_\ell(0_+) = 0$, we infer that the convex function f_ℓ is monotone non-decreasing on \mathbb{R}_+ . \square

Remark 3.12 (*growth conditions*). The inequality in (3.14) confirms that f_ℓ grows at least quadratically at infinity, as announced in the introduction (see Theorem 3.3). More specifically, we can highlight two subcases for a cost ℓ satisfying $(H1) - (H3)$.

- a) $k_\ell := \sup_{\delta > 0} \ell_-(\delta)\delta^d = +\infty$. Then $\liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} = +\infty$ and f_ℓ has a super quadratic growth. Note that this conclusion is consistent with the case $\ell(r) = r^{-s}$ for $s > d$ (see the next subsection).
- b) $\int_0^\infty \ell_+(t)t^{d-1} dt < +\infty$. In this case $k_\ell < +\infty$ and, thanks to (3.13) and to the estimate given in Lemma 2.7 (that we can apply to ℓ_+ with $r_0 = 0$), we obtain the lower and upper bounds:

$$0 < k_\ell \leq \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq \limsup_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq C \int_0^\infty \ell_+(t)t^{d-1} dt.$$

It follows that, under the integrability condition $\int_{\mathbb{R}^d} \ell_+(|x|) dx < +\infty$, f_ℓ enjoys a quadratic growth from above and from below.

3.4. Examples.

3.4.1. *The hard spheres model.* The hard spheres potential is given by

$$\ell(r) = \begin{cases} +\infty & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}$$

The computation of g_ℓ through (3.10) and (3.11) leads to a linear function on \mathbb{R}_+ namely $g_\ell(\lambda) = \gamma_d \lambda$, where γ_d denotes the densest spheres packing volume fraction in \mathbb{R}^d . This famous universal constant can be defined as

$$(3.15) \quad \gamma_d := \inf_{k \in \mathbb{N}^*} \frac{S(Q_k)}{k^d} = \lim_{k \rightarrow \infty} \frac{S(Q_k)}{k^d},$$

where, for any Borel set $A \subset \mathbb{R}^d$, $S(A)$ denotes the maximal number of points in A with mutual distance larger or equal to 1. The mean-field energy density f_ℓ given by Theorem 3.3 is therefore the indicator function of the interval $[0, \gamma_d]$

$$f_\ell(t) = 0 \quad \text{if } t \leq \gamma_d \quad , \quad f_\ell(t) = +\infty \quad \text{otherwise .}$$

Furthermore, for every continuous external potential $U \in \mathcal{C}(\overline{\Omega})$, we recover from Corollary 3.4 the convergence:

$$\min_{S \in \mathcal{F}_\varepsilon(\overline{\Omega})} \left\{ \varepsilon^d \sum_{z \in S} U(z) \right\} \rightarrow \gamma_d \int_{\overline{\Omega}} U(z) dz ,$$

where $\mathcal{F}_\varepsilon(\overline{\Omega})$ is the family of finite subsets $S \subset \overline{\Omega}$ satisfying $|x - y| \geq \varepsilon$ for all $(x, y) \in S^2 \setminus \Delta$.

Remark 3.13. A variant of the previous result was obtained recently in [3] in the case where the total number of particles N_ε is prescribed to satisfy $N_\varepsilon \varepsilon^d \rightarrow \kappa$ as $\varepsilon \rightarrow 0$ where $\kappa > 0$ is a given real parameter. With our notations this condition amounts to restrict the Γ -limit F to measures ρ such that $\int_{\Omega} u dx = \kappa$. Since the domain of F consists of density measures $\rho = u \mathcal{L}^d \llcorner \Omega$ such that $u \leq \gamma_d$ a.e., the latter integral condition requires that $\kappa \leq \gamma_d |\Omega|$ hence a congestion ratio $\theta := \frac{\kappa}{\gamma_d |\Omega|}$ not larger than 1. In this case and if, following the classical empirical measure representation, u is normalized to be a probability density by setting $\tilde{u} := \frac{u}{\int_{\Omega} u dx} = \frac{u}{\kappa}$, we recover a mean-field energy vanishing for $\tilde{u} \leq \frac{\gamma_d}{\kappa}$ and infinite otherwise, exactly as stated in [3, Thm 6.1]. Note that the duality technique used there could only handle cost functions ℓ taking values in the discrete set $\{0, +\infty\}$.

3.4.2. *The case of Riesz potentials.* Short range potential of Riesz type corresponds to fixing $s > d$ and taking

$$l(r) = r^{-s} \text{ on } \mathbb{R}_+^d .$$

In this case, it's easy to establish from the homogeneity of the cost ℓ that the bivariate function Γ_ℓ defined in (3.7) satisfies, for every $t \geq 0$, the following scaling law :

$$(3.16) \quad \Gamma_\ell(t\lambda, B) = t \Gamma_\ell(\lambda, t^{1/s} B) .$$

It follows from (3.16) that:

$$\frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \frac{t \Gamma_\ell(\lambda, t^{1/s} Q_k)}{k^d} = t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} .$$

Sending $k \rightarrow \infty$ and applying (3.11) two times, we get:

$$g_\ell(t\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \lim_{k \rightarrow \infty} t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} = t^{1+d/s} g_\ell(\lambda) .$$

In virtue of the equality $f_\ell = g_\ell^*$, we deduce that:

$$(3.17) \quad f_\ell(t) = C(s, d) t^{1+s/d}$$

where $C(s, d) = f_\ell(1)$ is a universal constant. We thus recover the Γ -convergence result proved in [12, 13].

Remark 3.14. If we chose $\ell(r) = r^{-s}$ where $s < d$, then condition (H3) is violated, and the scaling defined in equation (1.8) that we used to define F_ε will result in an infinite Γ -limit. This means that $F(\rho)$ will be equal to $+\infty$ whenever ρ is not equal to zero, and $F(0)$ will be equal to zero. We can observe this when we consider a system of N_ε particles in $S_\varepsilon \subset \bar{\Omega}$ such that $\rho_\varepsilon := \rho_{S_\varepsilon}^\varepsilon$ converges weakly to ρ , and $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < +\infty$. Assuming $\rho \neq 0$, then we have $N_\varepsilon \sim \|\rho\| \varepsilon^{-d}$ as $\varepsilon \rightarrow 0$. Moreover, due to the power law property of ℓ , we can write:

$$(3.18) \quad F_\varepsilon(\rho_\varepsilon) = \varepsilon^{s+d} \xi_\ell(S_\varepsilon) \sim \frac{\|\rho\|^2}{\varepsilon^{d-s}} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2}.$$

As ℓ satisfies (1.3), the convergence result of the long range case holds (with $h_N = N^{-2}$, see [3], [20]). Therefore, based on (1.4), and given that the standard empirical measure linked to S_ε converges to $\hat{\rho} = \frac{\rho}{\|\rho\|} \in \mathcal{P}(\bar{\Omega})$, it follows that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2} \geq D_\ell(\hat{\rho}) > 0.$$

This contradicts (3.18) since $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < \infty$. Therefore $\rho = 0$.

3.4.3. The case of finite costs. Many examples of finite costs can be considered as, for instance, ℓ being a step function with compact support. Owing to Corollary 3.11, the effective convex integrand f_ℓ has a quadratic growth on \mathbb{R}_+ . The simplest one is the penalized version of the hard spheres potential defined by:

$$(3.19) \quad \ell(r) = \begin{cases} \frac{M}{2} & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad (M \text{ positive parameter}).$$

Applying the lower bound (3.13) with $\delta = 1$, we deduce that $f_\ell \geq h$ where

$$h(t) := \frac{M}{2} t(t-1)_+.$$

In turn this lower bound is optimal for integer values of t since, as proved below, f_ℓ coincides on \mathbb{R}_+ with the piecewise affine interpolation of h given by:

$$(3.20) \quad f_\ell(t) = h(k) + (t-k)(h(k+1) - h(k)) \quad \forall t \in [k, k+1], k \in \mathbb{N}.$$

Proof. Owing to (3.11), the Fenchel conjugate of f_ℓ is given by

$$g_\ell(\lambda) = \lim_{K \rightarrow +\infty} \frac{1}{K} \sup_{S \subset [0, K]} \left\{ \lambda \#(S) - \frac{M}{2} \#(\{(x, y) \in S^2 \setminus \Delta_1\}) \right\},$$

where $\Delta_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \geq 1\}$. Let $S \subset [0, K]$ be an optimal set which we split in K disjoint pieces namely $S = \bigcup_{i=1}^K S_i$ where $S_i = S \cap [i, i + 1)$. Let us denote n_i the number of points in S_i . By pushing them to the center of the interval $[i, i + 1)$, we see that the number of pairs in $S^2 \setminus \Delta_1$ decreases to $\sum_{i=1}^K n_i(n_i - 1)$. It follows that:

$$\begin{aligned} g_\ell(\lambda) &= \lim_{K \rightarrow \infty} \sup_{n_i \in \mathbb{N}} \frac{1}{K} \left\{ \lambda \sum_{i=1}^K n_i - \frac{M}{2} \sum_{i=1}^K n_i(n_i - 1) \right\} \\ &= \sup_{n \in \mathbb{N}} \left\{ \lambda n - \frac{M}{2} n(n - 1) \right\} = (h + \chi_{\mathbb{N}})^*(\lambda), \end{aligned}$$

where $\chi_{\mathbb{N}}$ denotes the indicator function of the integers. Therefore $f_\ell = (g_\ell)^*$ is nothing else but the convexification of $h + \chi_{\mathbb{N}}$ given by the interpolation formula (3.20). \square

As demonstrated above, optimal point configurations for a constant external potential are obtained by periodically grouping a suitable number of points. Therefore, optimal sets are essentially multisets (see Remark 3.7). It is probable that a similar phenomenon occurs in higher dimensions.

The situation will vary if we consider a non-monotonic step function, such as the following:

$$\ell(r) = 1 \text{ for } r \in [0, 1], \quad \ell(r) = 4 \text{ for } r \in (1, 2), \quad \ell(r) = 0 \text{ for } r \geq 2.$$

In this case, we anticipate that optimal configurations may be periodic, but associated with a non-uniform Voronoi tessellation, consisting of patterns of different sizes, as observed in the context of optimal location problem (see [5, Sec 3.4]).

4. PROOF OF THE MAIN THEOREM

First, we check the properness property (2.16) to make sure that the upper Γ -limit of F_ε is not trivial. To do this, it is sufficient to apply Lemma 2.9 by choosing $u_0 = a \mathbf{1}_\Omega$ for $a \in [0, r_0^{-d}]$. Next, by virtue of the equi-coercivity property of F_ε proved in Proposition 2.5 and of the Kuratowski compactness theorem (see Proposition 3.1 and the introductory comment of Subsection 3.2), we can find a sequence $\varepsilon_k \rightarrow 0$ and a weak* lower semicontinuous functional $F : \mathcal{M}_+(\Omega) \rightarrow [0, +\infty]$ such that $F_{\varepsilon_k} \xrightarrow{\Gamma} F$ as $k \rightarrow \infty$. Note that the limit F may a priori depend on the chosen sequence $\varepsilon_k \rightarrow 0$. Accordingly, we will complete the proof of Theorem 3.3 in two steps which are outlined below:

Step 1: we show that F is a local functional of the form

$$(4.1) \quad F(\rho) := \begin{cases} \int_\Omega j(x, u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

where $j : \Omega \times \mathbb{R} \rightarrow [0, +\infty]$ is a suitable convex normal integrand such that $j(\cdot, 0) = 0$ a.e. in Ω . As a consequence F is convex, weak* l.s.c. and coincides with its Fenchel biconjugate, i.e. :

$$F(\rho) = F^{**}(\rho) = \sup_{v \in \mathcal{C}(\overline{\Omega})} \left\{ \int v d\rho - F^*(v) \right\}.$$

Step 2: we identify the Fenchel conjugate F^* in terms of the convex function g_ℓ defined in(3.10), namely:

$$F^*(v) = \int_{\Omega} g_\ell(v) dx \quad , \quad \text{for every } v \in \mathcal{C}(\overline{\Omega}).$$

It follows that the limit functional F *does not depend on the sequence* (ε_k) . Also, since (3.14) the convex function $f_\ell = g_\ell^*$ has a superlinear growth at infinity, by applying a classical result on convex functionals on measures (see for instance [6]), we obtain the equalities $F = F^{**} = F_\ell$ where:

$$F_\ell(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

This will conclude the proof of the Γ -convergence of the whole sequence (F_ε) as stated in Theorem 3.3.

Proof of Step 1. Let $\rho \in \mathcal{M}_+(\overline{\Omega})$ be such that $F(\rho) < +\infty$. Then, there exists a recovering sequence $\rho_k \xrightarrow{*} \rho$ such that $\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = F(\rho) < +\infty$. By the assertion ii) of Proposition 2.5, we infer that ρ is an absolutely continuous measure. Accordingly, there exists a functional $J : L^1(\Omega) \rightarrow [0, +\infty]$ such that

$$F(\rho) := \begin{cases} J(u) & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

The following result will be crucial for deriving the integral representation and the convexity of J . Its delicate proof is postponed to the end of this section.

Lemma 4.1. *The functional $J : L^1(\Omega) \rightarrow [0, +\infty]$ defined above satisfies the following:*

- (i) J is weakly lower semicontinuous and satisfies $J(0) = 0$;
- (ii) The domain of J is a subset of $L^1(\Omega; \mathbb{R}_+)$ and $J(u \mathbf{1}_A) \leq J(u)$ holds for every $u \in L^1(\Omega, \mathbb{R}_+)$ and every Borel subset $A \subset \Omega$;
- (iii) It holds $J(u + v) = J(u) + J(v)$ whenever $uv = 0$ is satisfied almost everywhere in Ω .

In view of the assertions (i) and (iii) of Lemma 4.1 and since the Lebesgue measure on Ω is atomless, we may apply a classical integral representation (see for instance Hiai and Umegaki [14, 15] or the monograph [8]) according to which there exists a suitable *convex* normal

integrand j such that (4.1) holds. Moreover, as $J \geq 0$ and $J(0) = 0$, we have $j(\cdot, 0) = 0$ a.e. while, due to the assertion (ii), the integrand j satisfies $j(x, t) = +\infty$ if $t < 0$. \square

Proof of Step 2. From Step 1 and by a classical result on integral functionals (see for instance [6]), the Fenchel conjugate of F is given for every $v \in \mathcal{C}(\overline{\Omega})$ by:

$$F^*(v) = \sup_{u \in L^1(\Omega)} \left\{ \int_{\Omega} v u \, dx - \int_{\Omega} j(x, u(x)) \, dx \right\} = \int_{\Omega} j^*(x, v(x)) \, dx.$$

Obviously we may extend this equality to all functions $v \in L^\infty(\Omega)$. Noticing that $j^*(x, 0) = -\inf j(x, \cdot) = -j(x, 0) = 0$, we observe that, for every $\lambda \in \mathbb{R}$ and for every hypercube $Q(x_0, a) \subset \Omega$, we have

$$F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = \int_{Q(x_0, a)} j^*(x, \lambda) \, dx.$$

Next we claim that, for any such an hypercube $Q(x_0, a) \subset \Omega$, the following holds:

$$(4.2) \quad F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda)$$

Suppose that this claim is true. Then, by considering Lebesgue points of $j^*(\cdot, \lambda)$ for λ in a dense countable subset D of \mathbb{R} , we can find a Lebesgue negligible subset $N \subset \Omega$ such that $j^*(x, \lambda) = g_\ell(\lambda)$ for all $(x, \lambda) \in (\Omega \setminus N) \times D$. Thanks to the continuity of g_ℓ proved in Proposition 3.10 and to the convexity of $j^*(x, \cdot)$, the latter equality can be then extended to all $(x, \lambda) \in (\Omega \setminus N) \times \mathbb{R}$, so that we have $F^*(v) = \int_{\Omega} g_\ell(v) \, dx$ for every $v \in \mathcal{C}(\overline{\Omega})$. Hence the conclusion of Step 2 is reached and the proof of Theorem 3.3 is complete provided we can confirm (4.2).

We now focus on the proof of the equality (4.2). For $\lambda \leq 0$, this equality is trivial since $g_\ell(\lambda) = j^*(x, \lambda) = 0$. Next we observe that, for every $\lambda \geq 0$ and $Q(x_0, a) \subset \Omega$, we have:

$$\begin{aligned} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) &:= \sup \{ \lambda \rho(Q(x_0, a)) - F_{\varepsilon_k}(\rho) : \rho \in \mathcal{M}_+(\overline{\Omega}) \} \\ &= \varepsilon_k^d \sup_{S \subset \overline{\Omega}} \{ \lambda \#(S \cap Q(x_0, a)) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S \subset Q(x_0, a)} \{ \lambda \#(S) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S' \subset Q(x_0, \frac{a}{\varepsilon_k})} \{ \lambda \#(S') - \xi_\ell(S'/\varepsilon_k) \} \\ &= \varepsilon_k^d \Gamma_\ell(\lambda, Q(x_0, \frac{a}{\varepsilon_k})) \end{aligned}$$

where:

- to pass from the second to the third line, we substitute any competitor $S \subset \overline{\Omega}$ with $S \cap Q(x_0, a)$ which has larger energy;

- to pass from the third line to the two last lines, we set $S = S'/\varepsilon_k$ for going back from the ε_k - scaled energy (1.7) to the ground interaction energy ξ_ℓ and ultimately recover the set function Γ_ℓ defined in (3.7).

Therefore, thanks to (3.11), we can pass to the limit $k \rightarrow +\infty$ (the position of x_0 is irrelevant) and obtain the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda).$$

So, proving (4.2) reduces to checking the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = F^*(\lambda \mathbf{1}_{Q(x_0, a)}),$$

that we rewrite in the equivalent form:

$$(4.3) \quad \inf_{\rho \in \mathcal{M}_+(\overline{\Omega})} \{F_{\varepsilon_k}(\rho) - \lambda \rho(Q(x_0, a))\} \rightarrow \inf_{\rho} \{F(\rho) - \lambda \rho(Q(x_0, a))\}.$$

The left hand side infimum in (4.3) being non-positive (easily seen by taking $\rho = 0$ as a competitor), we may apply Proposition 2.5 with the choice $U = -\lambda \mathbf{1}_{Q(x_0, a)}$. Therefore any minimizing sequence (ρ_k) for the left hand side of (4.3) is bounded in $\mathcal{M}_+(\overline{\Omega})$ hence weakly* relatively compact. By the assertion iii) of Proposition 3.1, we will be able to conclude the convergence of infima in (4.3) if we can show that

$$(4.4) \quad F_{\varepsilon_k} + G \xrightarrow{\Gamma} F + G \quad \text{being} \quad G(\rho) := -\lambda \rho(Q(x_0, a)).$$

In virtue of Theorem 3.3, we already know that $F_{\varepsilon_k} \xrightarrow{\Gamma} F$. Then it is enough to invoke the stability property of the assertion iv) of Proposition 3.1. However the functional G given above is not weak* continuous on $\mathcal{M}(\overline{\Omega})$ and therefore, we need to verify the less stringent requirements set out in (3.2). The first one is satisfied since $|G(\rho)| \leq \lambda \|\rho\|$. For the second one, we observe that, if $F(\rho) < +\infty$, then ρ is of the form $\rho = u \mathcal{L}^d \llcorner \Omega$ (see Proposition 2.5), hence $\rho(\partial Q(x_0, a)) = 0$ and every sequence $\rho_n \xrightarrow{*} \rho$ satisfies $\rho_n(Q(x_0, a)) \rightarrow \rho(Q(x_0, a)) = \int_{Q(x_0, a)} u dx$. This confirms the validity of (4.4), hence that of (4.3). As a result the equality (4.2) is proved and, as announced, this achieves Step 2 and the proof of Theorem 3.3. \square

Proof of Lemma 4.1. (i) Let $u_k \rightarrow u$ in $L^1(\Omega)$. Then $\rho_k = u_k \mathcal{L}^d \llcorner \Omega$ and $\rho = u \mathcal{L}^d \llcorner \Omega$ are such that $\rho_k \xrightarrow{*} \rho$ in $\mathcal{M}_+(\overline{\Omega})$. Since $F = \Gamma\text{-}\lim F_{\varepsilon_k}$ is weak* lower semicontinuous, we infer that

$$\liminf_{k \rightarrow \infty} J(\overline{u}_k) = \liminf_{k \rightarrow \infty} F(\rho_k) \geq F(\rho) = J(u).$$

To show that $J(0) = 0$, we consider as S_k a singleton $\{x_0\}$ so that the associated measure $\rho_k = (\varepsilon_k)^d \delta_{x_0}$ satisfies $\rho_k \xrightarrow{*} 0$ while $F_{\varepsilon_k}(\rho_k) = 0$.

(ii) By the definition of the Γ -limit, $F(\rho) < +\infty$ implies that ρ is a weak* limit of a sequence (ρ_k) in $\mathcal{M}_+(\overline{\Omega})$, hence of the form $\rho = u \mathcal{L}^d \llcorner \Omega$

with $u \geq 0$. Let now $u \in L^1(\Omega; \mathbb{R}_+)$ and A a Borel subset of Ω . We show first that $J(u1_A) \leq J(u)$ if $\mathcal{L}^d(\partial A) = 0$. We may assume that $J(u) < +\infty$. Hence, there exists a family of subsets $S_k \subset \Omega$ such that $\rho_k = \mu_{S_k}^{\varepsilon_k}$ satisfies $\rho_k \xrightarrow{*} \rho := u \mathcal{L}^d \llcorner \Omega$ and $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$. If we let $S'_k = S_k \cap A$ and $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$, we have $\rho'_k \xrightarrow{*} u1_A \mathcal{L}^d \llcorner \Omega$, since indeed the convergence $\rho_k \xrightarrow{*} \rho$ is tight while $\rho(\partial A) = 0$. Therefore, $F_{\varepsilon_k}(\rho'_k) \leq F_{\varepsilon_k}(\rho_k)$. By passing to the limit, as $k \rightarrow +\infty$, we deduce that

$$J(u1_A) \leq F(\rho1_A) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = J(u).$$

To extend the inequality to any Borel subset A of Ω , it is enough to consider an approximating sequence (A_n) such that

$$\mathcal{L}^d(\partial A_n) = 0 \quad , \quad \mathcal{L}^d(A_n \Delta A) \rightarrow 0,$$

and then pass to the limit in the inequality $J(u) \geq J(u1_{A_n})$ while letting $n \rightarrow +\infty$. Indeed, the conclusion will then follow from the lower semicontinuity of J with respect to the norm convergence in $L^1(\Omega)$. Now, to construct such a sequence (A_n) , we consider a compact subset $K_n \subset A$ and an open subset $\omega_n \supset A$ such that $\mathcal{L}^d(\omega_n \setminus K_n) \leq \frac{1}{n}$. For every n , we can choose a suitable $r_n > 0$ such that the enlarged open set $A_n = K_n + B(0, r_n)$ satisfies $\mathcal{L}^d(\partial A_n) = 0$ ³ while $A_n \subset \omega_n$. Then clearly $\mathcal{L}^d(A_n \Delta A) \leq \mathcal{L}^d(\omega_n \setminus K_n) \rightarrow 0$.

Let us now prove now the assertion (iii). In a first step, we assume that $\text{spt}(u) \cap \text{spt}(v) = \emptyset$ so that there exists open subsets $A \supset \text{spt}(u)$ and $B \supset \text{spt}(v)$ such that $\text{dist}(A, B) := \eta > 0$.

We begin by proving the inequality $J(u+v) \geq J(u) + J(v)$. Without loss of generality, we may assume that $J(u+v) < +\infty$ (hence u and v are non-negative). Then there exists a sequence of sets $S_k \subset \bar{\Omega}$ such that

$$\rho_k = \mu_{S_k}^{\varepsilon_k} \xrightarrow{*} (u+v) \mathcal{L}^d \llcorner \Omega \quad , \quad F_{\varepsilon_k}(\rho_k) \rightarrow J(u+v).$$

We write $S_k = S'_k \cup S''_k$ where $S'_k = S_k \cap A$ and $S''_k = S_k \cap B$ are disjoint. Then, we have $\rho_k = \rho'_k + \rho''_k$ where $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$ and $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$. Clearly $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$ while $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$. Therefore, by applying the Γ -lim inf inequality to ρ'_k and ρ''_k while taking into account the super-additivity property of F_{ε_k} (see (2.1)), we deduce that:

$$\begin{aligned} J(u+v) &= \lim_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\geq F(u \mathcal{L}^d \llcorner \Omega) + F(v \mathcal{L}^d \llcorner \Omega) = J(u) + J(v). \end{aligned}$$

To show the converse inequality $J(u+v) \leq J(u) + J(v)$, we assume without any loss of generality that $J(u) < +\infty$ and $J(v) < +\infty$. Then

³Here we use the fact that the function $\alpha_n(r) = \mathcal{L}^d(\{x \in \Omega : \text{dist}(x, K_n) > r\})$ is bounded monotone non increasing so that it is continuous except possibly on a finite or countable subset of \mathbb{R}_+ .

we consider recovering sequences $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$ and $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$ such that $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$, $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$ and $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$, $F_{\varepsilon_k}(\rho''_k) \rightarrow J(v)$. Up to dropping the elements of S'_k which are not in A and the elements of S''_k which are not in B , we may assume that $\text{spt}(\rho'_k) \subset A$ and $\text{spt}(\rho''_k) \subset B$. Indeed, removing these points will not affect the weak* convergence to $u \mathcal{L}^d \llcorner \Omega$ and $v \mathcal{L}^d \llcorner \Omega$ respectively while the total energy ξ_{ε_k} will not increase. Therefore, by exploiting the right hand inequality in (2.1) and since $\rho_k = \rho'_k + \rho''_k$ converge weakly* to $\rho = (u + v) \mathcal{L}^d \llcorner \Omega$, we are led to the following set of inequalities:

$$\begin{aligned} J(u + v) &\leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\quad + \limsup_{k \rightarrow \infty} \frac{2\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d} \|u\|_{L^1(\Omega)} \|v\|_{L^1(\Omega)} \\ &\leq J(u) + J(v) + C \limsup_{k \rightarrow \infty} \frac{\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d}, \end{aligned}$$

where, in the second line, we used the tight convergence of ρ'_k, ρ''_k . The conclusion follows by noticing that ℓ_+ coincides with ℓ for large values where it is non-increasing. Thus the integrability condition (H3) implies that $r^d \ell(r) \rightarrow 0$ as $r \rightarrow +\infty$. The desired sub-additivity inequality follows.

In a second step, we remove the strict separation condition on the supports of u and v by simply assuming that the upper-level sets $A := \{u > 0\}$ and $B := \{v > 0\}$ satisfy $\mathcal{L}^d(A \cap B) = 0$ (which, for u, v non-negative, is equivalent to say that $uv = 0$ a.e.). To that aim, possibly after substituting A, B with non-intersecting Borel representatives, we consider increasing sequences of compact subsets $K'_n \subset A, K''_n \subset B$ such that:

$$\mathcal{L}^d(A \setminus K'_n) \rightarrow 0, \quad \mathcal{L}^d(B \setminus K''_n) \rightarrow 0, \quad K'_n \cap K''_n = \emptyset.$$

Then $u_n := u \mathbf{1}_{K'_n}, v_n := v \mathbf{1}_{K''_n}$ satisfy $J(u_n + v_n) = J(u_n) + J(v_n)$. In virtue of assertion (ii), we infer that, for every n :

$$J(u_n + v_n) \leq J(u) + J(v) \quad \text{and} \quad J(u_n) + J(v_n) \leq J(u + v).$$

In virtue of the lower semicontinuity of J , since $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L^1(\Omega)$, we deduce from above the inequalities $J(u + v) \leq J(u) + J(v)$ and $J(u) + J(v) \leq J(u + v)$, hence the desired additivity property. \square

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