



HAL
open science

Mean field theory for a general class of short-range interaction functionals

Guy Bouchitté, Rajesh Mahadevan

► **To cite this version:**

Guy Bouchitté, Rajesh Mahadevan. Mean field theory for a general class of short-range interaction functionals. 2023. hal-04257842

HAL Id: hal-04257842

<https://univ-tln.hal.science/hal-04257842>

Preprint submitted on 25 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

MEAN FIELD THEORY FOR A GENERAL CLASS OF SHORT-RANGE INTERACTION FUNCTIONALS

GUY BOUCHITTÉ AND RAJESH MAHADEVAN

ABSTRACT. In models of N interacting particles in \mathbb{R}^d as in Density Functional Theory or crowd motion, the repulsive cost is usually described by a two-point function $c_\varepsilon(x, y) = \ell\left(\frac{|x-y|}{\varepsilon}\right)$ where $\ell : \mathbb{R}_+ \rightarrow [0, \infty]$ is decreasing to zero at infinity and parameter $\varepsilon > 0$ scales the interaction distance. In this paper we identify the mean-field energy of such a model in the short-range regime $\varepsilon \ll 1$ under the sole assumption that $\exists r_0 > 0 : \int_{r_0}^\infty \ell(r)r^{d-1} dr < +\infty$. This extends recent results [12, 13, 17] obtained in the homogeneous case $\ell(r) = r^{-s}$ where $s > d$.

Keywords: empirical measures, non-local functionals, Γ -convergence, mean-field energy, sub-additivity

Mathematics Subject Classification: 49J45, 49K21, 49N15, 60B10, 70-10, 82B21

1. INTRODUCTION

We consider a repulsive interaction function on $(\mathbb{R}^d)^N$ of the kind

$$(1.1) \quad c_N^\varepsilon(x_1, \dots, x_N) = \sum_{i \neq j} \ell\left(\frac{|x_i - x_j|}{\varepsilon}\right).$$

where:

- N is the number of particles in \mathbb{R}^d ;
- $\varepsilon > 0$ scales the interaction distance between particles.
- the two-particle cost $\ell : [0, +\infty] \rightarrow [0 + \infty]$ satisfies:

(H1) ℓ is l.s.c. and $\ell(0) > 0$ ($\ell(0) = +\infty$ is allowed)

(H2) $\exists r_0 \geq 0$ such that ℓ is finite and non increasing on $[r_0, +\infty)$ and $\lim_{r \rightarrow \infty} \ell(r) = 0$.

Date: October 24, 2023.

1991 Mathematics Subject Classification. 00X99.

Part of this work was prepared at FCFM, Univ. Concepci'ón. The first author warmly thanks this institution for its financial support and hospitality during the fall of 2022.

The second author acknowledges the support of Direcci'ón de Postgrado, UdeC for the funding received through the project UCO 1866 .

In the whole paper, we denote by $\overline{\Omega}$ the closure of a smooth domain $\Omega \subset \mathbb{R}^d$ where the N particles are located. We speak of a *confined system* when $\overline{\Omega}$ is compact (container). Given a continuous exterior potential $U : \overline{\Omega} \rightarrow \mathbb{R}$, we consider the finite dimensional problem:

$$(1.2) \quad \mathcal{E}_N^\varepsilon(\Omega, U) := \inf \left\{ h_N c_N^\varepsilon(x_1, \dots, x_N) + \frac{1}{N} \sum_{i=1}^N U(x_i) : x_i \in \overline{\Omega} \right\},$$

where h_N is a suitable chosen normalization factor. Since the seminal work of Choquet in 1958 [9] and the growing interest of the quantum and statistical mechanics community, a lot of work has been devoted to the limit behavior of $\mathcal{E}_N^\varepsilon(\Omega, U)$ as $N \rightarrow \infty$ (ε fixed) as well as the characterization of the weak cluster points of the empirical measures associated with N -point configurations of minimal energy. The cornerstone of the mean field theory consists in identifying a limit energy functional on measures whose minimizers are precisely these cluster points.

1.1. State of the art. The scaling factor h_N in (1.2) must be selected so that the limit of the infimum belongs to $(0, +\infty)$. In turn this issue relies heavily on the integrability properties of the function $g(x) = \ell(|x|)$. Let us report on two cases of major interest:

1.1.1. Long range interaction case. Here $\Omega = \mathbb{R}^d$ and we take $\varepsilon = 1$. Moreover in addition to (H1)(H2), we assume that $g \in L_{loc}^1(\mathbb{R}^d)$ i.e.:

$$(1.3) \quad \int_0^1 r^{d-1} \ell(r) dr < +\infty.$$

In that case, a relevant choice is $h_N = \frac{1}{N^2}$ meaning roughly that the interaction energy is averaged over all pairs of distinct points in $\{x_1, x_2, \dots, x_N\}$. The identification of the mean-field energy is well known in the case of Riesz potentials $\ell(r) = \frac{1}{r^s}$ for $0 < s < d$, in the Logarithmic case $\ell(r) = -\log(r)$ for $d = 2$ and more generally for ℓ of positive type i.e. such that the Fourier transform of $g(x) = \ell(|x|)$ is positive in \mathbb{R}^d (see for instance the monograph by S. Serfaty [20]). It is given by a non-local functional, the so called *Direct energy*:

$$(1.4) \quad D_\ell(\rho) := \iint \ell(|x - y|) \rho \otimes \rho(dx dy).$$

Accordingly the limit problem associated with (1.2) reads:

$$\mathcal{E}_\infty(\Omega, U) = \inf \left\{ D_\ell(\rho) + \int U d\rho : \rho \in \mathcal{P}(\mathbb{R}^d) \right\},$$

where the infimum is reached at a unique configuration provided U grows suitably at infinity. At this stage, a few comments are in order:

- in the case of a confining external potential U , there are several impressive works devoted to the next order asymptotics [20, 19, 21, 10] in the case of Riesz potentials $\ell(r) = r^{-s}$ for $d \geq 3$ and $d - 2 \leq s < d$ revealing an asymptotic behavior as $N \rightarrow \infty$ of the form:

$$(1.5) \quad \lim_{N \rightarrow \infty} N^{1-\frac{s}{d}} (\mathcal{E}_N(\Omega, U) - \mathcal{E}_\infty(\Omega, U)) = C(s, d) \int (\rho_U)^{1+\frac{s}{d}},$$

where ρ_U is the unique minimizer realizing $\mathcal{E}_\infty(\Omega, U)$.

- if U remains bounded at infinity (for instance a Coulomb potential vanishing at infinity), the existence of a minimizer ρ_U may fail due to a loss of mass at infinity along minimizing sequences. A relaxation procedure leads to consider minimizers in the class of sub-probabilities $\rho \in \mathcal{P}_-(\mathbb{R}^d)$ and involves the weak* lower semicontinuous convexification of the Direct energy D . If ℓ is of positive type, this relaxed energy coincides with the natural 2-homogeneous extension of D to $\mathcal{P}_-(\mathbb{R}^d)$ while almost nothing is known if ℓ is merely locally integrable. For further details and examples of relaxed minimizers, we refer to the recent paper [3].

1.1.2. *Short range interaction case.* Following an idea developed for the hard spheres model [3], we look now at ε as a small parameter tending to zero with a prescribed speed as $N \rightarrow \infty$. If one thinks to a container $\bar{\Omega}$ of unit volume and ε to be the average distance of a particle to the others, it is natural to consider an asymptotic analysis where the product $\varepsilon^d N$ remains constant or converges to a given intensity factor $\kappa \in (0, +\infty)$. In a crowd model, this factor κ is related to a congestion ratio (see the hard spheres model in Section 3.4 and Remark 3.13). This of course means that we need to assume that $\varepsilon \sim N^{-\frac{1}{d}}$. Accordingly, in order to obtain a precise scaling for h_N ensuring a non-trivial behavior of the infimum (1.2), it is crucial to make an additional integrability on ℓ at infinity namely

$$(H3) \quad \int_{r_0}^{+\infty} \ell(r) r^{d-1} dr < +\infty.$$

It turns out that, under (H3), the right scaling factor in (1.2) is $h_N = \frac{1}{N}$ in contrast with the long range case. This covers the case of hyper singular Riesz potentials $\ell(r) = r^{-s}$ with $s > d$. For such potentials the parameter ε can be dropped thanks to the homogeneity and the normalized interaction energy becomes $\frac{1}{N^{1+\frac{s}{d}}} c_N^1(x_1, \dots, x_N)$. Under the latter scaling, it was proved recently [12, 13] that the mean field energy is a local functional defined on absolutely continuous measures $\rho = u \mathcal{L}^d \llcorner \Omega$ by $F(\rho) = C(s, d) \int_\Omega u^{1+\frac{s}{d}} d\mathcal{L}^d$, being $C(s, d)$ a universal constant. However, extending this result to more general costs seems to be difficult in the framework developed in [12], except possibly if ℓ is very close to a power potential.

1.2. Our contribution. This paper proposes a significant simplification of asymptotic analysis in the short range case. The approach is based on two components: first, treating the *interaction distance* ε in (1.1) as an infinitesimal parameter, and second, using an ε -counterpart of the traditional empirical measure frequently utilized in mean-field theory.

Thus, for every cost ℓ satisfying (H3), we can determine the mean field energy in terms of a *local* integral functional of the type $\int_{\Omega} f_{\ell}(u) dx$, where $u = \frac{d\rho}{dx}$ denotes the local particle density and f_{ℓ} is a convex integrand that exhibits super-linear growth at infinity. This expands upon previous findings [3] that were limited to the hard-spheres model (where $\ell(r) = +\infty$ when $r < 1$ and $\ell(r) = 0$ otherwise). Similarly, this permits to handle the case of hyper-singular Riesz potentials $\ell(r) = r^{-s}$ for $s > d$ analyzed in [12, 13].

It is noteworthy that fulfilling the integrability condition (H3) is crucial and cannot be sidestepped. When assuming that Ω is bounded, a cost ℓ that satisfies $\int_{r_0}^{+\infty} r^{d-1} \ell(r) dr = +\infty$ would give rise to an infinite limit in (1.2) if the scaling is by $h_N = \frac{1}{N}$ and $\varepsilon^d N \sim 1$ (see Remark 3.14).

1.3. Setting of the asymptotic problem and notations. From now on, Ω will be a *bounded* domain of \mathbb{R}^d with Lipschitz boundary ($\partial\Omega$ needs to be \mathcal{L}^d -negligible) and we consider a cost function ℓ which satisfies the standing assumptions (H1), (H2) and (H3).

For purposes of presentation, we utilize the infinitesimal length ε as the main parameter while the number of particles $N = N_{\varepsilon}$ approaches infinity as ε tends towards zero, following the scale $N_{\varepsilon} \sim \kappa \varepsilon^{-d}$ where κ is a positive constant. Later, we will establish that assigning a value to κ is not required because a uniform bound on the N_{ε} -point interaction energy will automatically result in $\limsup_{\varepsilon \rightarrow 0} N_{\varepsilon} \varepsilon^d < +\infty$. We can now

incorporate measures in $\overline{\Omega}$ into a variational framework for addressing the mean field problem. For each finite subset $S \subset \overline{\Omega}$, we define its ε -scaled empirical measure as follows:

$$(1.6) \quad \rho_S^{\varepsilon} := \varepsilon^d \sum_{x \in S} \delta_x.$$

This measure belongs to the set of non-negative Borel measures on $\overline{\Omega}$, denoted by $\mathcal{M}_+(\overline{\Omega})$. Here, $\|\rho\|$ represents the total mass, which may be infinite, of any element $\rho \in \mathcal{M}_+(\overline{\Omega})$. Through this, we can observe that $\|\rho_S^{\varepsilon}\| = \varepsilon^d \#(S)$, which could deviate from the classical empirical measure of S_{ε} with a total mass equal to 1. A key advantage of this approach is that it allows to avoid the non-local constraint that all competitors must belong to the subclass $\mathcal{P}(\overline{\Omega})$ of probability measures.

Next, we define the ε -scaled interaction energy of a discrete set $S \subset \mathbb{R}^d$ as follows:

$$(1.7) \quad \xi_{\ell,\varepsilon}(S) = \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left(\frac{|x-y|}{\varepsilon} \right) \quad \text{where} \quad \Delta := \{(x,x) : x \in \mathbb{R}^d\}.$$

We will refer to the interaction energy corresponding to $\varepsilon = 1$ as the “ground interaction energy”, denoted by ξ_ℓ . When the cost function ℓ is fixed, we will use ξ_ε instead of $\xi_{\ell,\varepsilon}$. With this in mind, we can define a scaled energy functional $F_\varepsilon : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$ for every $\varepsilon > 0$ in the following way:

$$(1.8) \quad F_\varepsilon(\rho) = \begin{cases} \varepsilon^d \xi_\varepsilon(S) & \text{if } \exists S \subset \overline{\Omega} \text{ such that } \rho = \rho_S^\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

The discrete problem (1.2) for $h_N = \frac{1}{N}$ and $N \varepsilon^d \sim \kappa$ can then be expressed through the relation:

$$\kappa \mathcal{E}_N^\varepsilon(\Omega, U) \sim \inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho : \rho \in \mathcal{M}_+(\overline{\Omega}) \right\} \quad \text{as } \varepsilon \rightarrow 0.$$

Accordingly the mean-field energy will be represented by a functional $F : \mathcal{M}_+(\overline{\Omega}) \rightarrow [0, +\infty]$ characterized by the property that, for every $U \in \mathcal{C}(\overline{\Omega})$, one has the convergence of infima

$$\inf \left\{ F_\varepsilon(\rho) + \int_{\overline{\Omega}} U d\rho \right\} \rightarrow \inf \left\{ F(\rho) + \int_{\overline{\Omega}} U d\rho \right\}$$

accompanied by the tight convergence of minimizers. This falls squarely within the Γ -convergence theory ([2, 11, 7]) on which we rely to support our results.

The paper is organized as follows: in Section 2, we establish a lower bound which allows to obtain the strong equi-coercivity of the sequence (F_ε) ; in addition we show that any weak* cluster point of a sequence (ρ_ε) with uniformly bounded energy is absolutely continuous with respect to the Lebesgue measure; in Section 3, we state the Γ -convergence of F_ε as $\varepsilon \rightarrow 0$ to a convex functional of the form $F(\rho) = \int_{\overline{\Omega}} f_\ell \left(\frac{d\rho}{dx} \right) dx$, where the effective integrand f_ℓ grows at least quadratically at infinity. It is given by the thermodynamical limit of a subadditive set function (*Krengel's theorem*). Some examples and applications are given. The Section 4 is devoted to the proof of the main theorem.

Notations:

- $B(x, r)$ is the open ball of the Euclidean space \mathbb{R}^d centered at x and of radius r ; if $x = 0$, we simply denote B_r ;
- Q_k denotes the hypercube $[-k/2, k/2]^d$, $Q(x_0, r) := x_0 + r Q_1$;
- $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$ stands for the diagonal of \mathbb{R}^d ;

- $\sharp(S)$ denotes the counting measure of a subset $S \subset \mathbb{R}^d$ ($+\infty$ if S is infinite);
- \mathcal{L}^d is the Lebesgue measure in \mathbb{R}^d ; given any Borel set B , $|B|$ is a short notation for $\mathcal{L}^d(B)$; ω_d is such that $|B(x, r)| = \omega_d r^d$;
- $\mathcal{C}(\overline{\Omega})$ denotes the Banach space of continuous functions on the compact subset $\overline{\Omega}$ equipped with the uniform norm;
- $\mathcal{M}(\overline{\Omega})$ stands for the space of signed Radon measures on $\overline{\Omega}$ equipped with the total variation norm;
- $\mathcal{P}_-(\overline{\Omega})$ (resp. $\mathcal{P}(\overline{\Omega})$) is the subset of Borel measures $\mu \in \mathcal{M}_+(\overline{\Omega})$ such that $\|\mu\| := \mu(\overline{\Omega}) \leq 1$ (resp. $\|\mu\| = 1$).
- The topological support of $\mu \in \mathcal{M}_+(\overline{\Omega})$ is denoted $\text{supp}(\mu)$ while $\mu \llcorner A$ represents its restriction to a Borel subset $A \subset \overline{\Omega}$;
- The bracket $\langle \cdot, \cdot \rangle$ will denote the duality between $\mathcal{C}(\overline{\Omega})$ and $\mathcal{M}(\overline{\Omega})$:

$$\langle v, \mu \rangle = \int v d\mu,$$

This duality induces the weak* topology on $\mathcal{M}(\overline{\Omega})$ which can be identified with the dual of $\mathcal{C}(\overline{\Omega})$; as $\overline{\Omega}$ is compact, the weak* convergence $\mu_h \xrightarrow{*} \mu$ in $\mathcal{M}_+(\overline{\Omega})$ implies the tight convergence since $\|\mu_h\| = \langle 1, \mu_h \rangle \rightarrow \langle 1, \mu \rangle = \|\mu\|$.

- To any non-empty set A , we associate the functions:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}, \quad \chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}.$$

2. ENERGY ESTIMATES AND COMPACTNESS.

We begin with some elementary properties of the set function $\xi_\ell(S)$ (defined by (1.7) for $\varepsilon = 1$).

Lemma 2.1. *Let S_1, S_2 be finite disjoint subsets of \mathbb{R}^d . Then we have:*

(i) (super-additivity)

$$\xi_\ell(S_1 \cup S_2) \geq \xi_\ell(S_1) + \xi_\ell(S_2).$$

(ii) (sub-additivity at large distance)

$$\xi_\ell(S_1 \cup S_2) \leq \xi_\ell(S_1) + \xi_\ell(S_2) + 2\ell_+(\eta) \sharp(S_1) \sharp(S_2),$$

where $\eta := \text{dist}(S_1, S_2)$.

Proof. Since S_1 and S_2 are non-intersecting, we can split $(S_1 \cup S_2)^2$ in four disjoint pieces as follows:

$$(S_1 \cup S_2)^2 = (S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_1 \times S_2) \cup (S_2 \times S_1).$$

The inequality (i) is then straightforward whereas, for the (ii), we simply majorize by $\ell_+(\eta)$ the contribution $\ell(|x - y|)$ of each pair (x, y) in $(S_1 \times S_2) \cup (S_2 \times S_1)$ where $|x - y| \geq \eta$ holds. \square

Recalling the notations (1.6) and (1.8) given in the introduction, we may rewrite a ε - rescaled version of Lemma 2.1 as follows: for every $\varepsilon > 0$ and all pairs of non-intersecting subsets $(S'_\varepsilon, S''_\varepsilon)$, one has

$$(2.1) \quad F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon + \rho''_\varepsilon) \leq F_\varepsilon(\rho'_\varepsilon) + F_\varepsilon(\rho''_\varepsilon) + \frac{2\ell_+(\eta_\varepsilon)}{\varepsilon^d} \|\rho'_\varepsilon\| \|\rho''_\varepsilon\|,$$

where $\rho'_\varepsilon := \rho_{S'_\varepsilon}^\varepsilon$, $\rho''_\varepsilon := \rho_{S''_\varepsilon}^\varepsilon$ and $\eta_\varepsilon = \varepsilon^{-1} \text{dist}(S'_\varepsilon, S''_\varepsilon)$.

2.1. A fundamental lower-bound. Since we are only assuming that $\ell(r)$ is decreasing for suitably large r , we need to define :

$$(2.2) \quad \ell_+(r) := \sup \{ \ell(s) : s \geq r \} \quad , \quad \ell_-(r) := \inf_{x,y \in [0,r]^d} \ell(|x-y|)$$

which are monotone non-increasing and satisfy:

$$\ell_+(r) = \ell(r) \quad \forall r \geq r_0 \quad , \quad \ell_-(r) = \inf \left\{ \ell(s) : s \leq r\sqrt{d} \right\} \leq \ell_+(r\sqrt{d}).$$

Next, in the same line as in the survey [17], we derive a very simple but fundamental lower bound for $\xi_\ell(S)$ when S is a N -point system contained in a Borel subset B of finite volume in \mathbb{R}^d . For such a B and any $\delta > 0$, we denote by $m_\delta(B)$ the minimal number of disjoint δ -hypercubes $Q_j = x_j + [-\delta/2, \delta/2]^d$ such that $B \subset \cup_{j=1}^{m_\delta(B)} \overline{Q_j}$. Then it's easy to check that $m_\delta(B) \sim \delta^{-d} \mathcal{L}^d(B)$ as $\delta \rightarrow 0$. For such subsets B , we will often use the following equivalent version obtained by keeping δ fixed while using large dilations:

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^d m_\delta \left(\frac{B}{\varepsilon} \right) = \delta^{-d} \mathcal{L}^d(B).$$

Lemma 2.2. *Let $B \subset \mathbb{R}^d$ be a Borel subset such that $\mathcal{L}^d(B) < +\infty$. Then for any N -point system $S \subset B$ and any $\delta > 0$, we have*

$$(2.4) \quad \xi_\ell(S) \geq N \ell_-(\delta) (\zeta - 1)_+ \quad \text{where} \quad \zeta = \frac{N}{m_\delta(B)}.$$

Proof. Let $\{Q_j, 1 \leq j \leq m_\delta(B)\}$ be a covering of B by disjoint δ -hypercubes Q_j and denote $n_j = \#(S \cap Q_j)$. We have $N = \sum n_j$ while, by the super additivity of ξ_ℓ (see Lemma 2.1) and the definition of $\ell_-(\delta)$, we have:

$$\xi_\ell(S) \geq \sum_{j=1}^{m_\delta(B)} \xi_\ell(S \cap Q_j) \geq \ell_-(\delta) \sum_{j=1}^{m_\delta(B)} n_j(n_j - 1).$$

If $N > m_\delta(B)$ (i.e. $\zeta > 1$), the desired lower bound (2.4) follows by noticing that the infimum

$$\inf \left\{ \sum_{j=1}^{m_\delta(B)} t_j(t_j - 1) : \sum_{j=1}^{m_\delta(B)} t_j = N, t_j \in \mathbb{R} \right\}$$

is reached for $t_j = \frac{N}{m_\delta(B)} = \zeta, \forall j$.

If $\zeta \leq 1$ the inequality (2.4) is trivial since $\xi_\ell(S) \geq 0$. \square

Remark 2.3. By the lower semicontinuity assumption (H1), since $\ell_-(0) = \ell(0) \in (0, +\infty]$, we may always find a $\delta > 0$ such that $\ell_-(\delta) > 0$. Note that the inequality (2.4) is still valid if $\ell_-(\delta) = +\infty$ provided we agree that $0 \times +\infty = 0$ (this situation occurs when $\ell = +\infty$ on an interval $[0, \delta_0)$).

2.2. Strong coercivity and compactness. We recall the definition of the functional F_ε given in (1.8) whose domain consists of ε -empirical measures of discrete subsets $S_\varepsilon \subset \overline{\Omega}$ (see the definition (1.6)).

Lemma 2.4 (strong coercivity). *Assuming that ℓ satisfies (H1) (H2), let $\delta > 0$ be such that $\ell_-(\delta) \in (0, +\infty]$ and set $\alpha := \frac{\delta^d \ell_-(\delta)}{\mathcal{L}^d(\Omega)}$.*

Then we have:

$$(2.5) \quad \liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|^2} \geq \alpha,$$

whenever (ρ_ε) is a sequence such that $\|\rho_\varepsilon\| \rightarrow +\infty$.

Proof. Without loss of generality, we may assume that $F_\varepsilon(\rho_\varepsilon) < +\infty$. Thus $\rho_\varepsilon = \rho_{S_\varepsilon}^\varepsilon$ for a suitable N_ε -point system $S_\varepsilon \subset \overline{\Omega}$ while:

$$F_\varepsilon(\rho) = \varepsilon^d \xi_\varepsilon(S_\varepsilon) = \varepsilon^d \xi_\ell \left(\frac{S_\varepsilon}{\varepsilon} \right), \quad \|\rho_\varepsilon\| = N_\varepsilon \varepsilon^d.$$

By applying the lower bound (2.4) to the subset $\varepsilon^{-1}S \subset \varepsilon^{-1}\overline{\Omega}$, we get:

$$(2.6) \quad F_\varepsilon(\rho_\varepsilon) \geq \ell_-(\delta) \|\rho_\varepsilon\| \left(\frac{\|\rho_\varepsilon\|}{\beta_\varepsilon} - 1 \right)_+ \quad \text{where } \beta_\varepsilon = \varepsilon^d m_\delta \left(\frac{\overline{\Omega}}{\varepsilon} \right).$$

From (2.3), we know that $\beta_\varepsilon \rightarrow \delta^{-d} \mathcal{L}^d(\Omega)$. Hence (2.5) follows by dividing (2.6) by $\|\rho_\varepsilon\|^2$ and passing to the limit as $\varepsilon \rightarrow 0$. \square

Proposition 2.5 (compactness). *Assuming that ℓ satisfies (H1) and (H2), let $U : \overline{\Omega} \rightarrow \mathbb{R}$ be a bounded Borel function and (ρ_ε) a sequence in $\mathcal{M}_+(\overline{\Omega})$ such that*

$$(2.7) \quad \sup_\varepsilon \left(F_\varepsilon(\rho_\varepsilon) + \int U d\rho_\varepsilon \right) < +\infty.$$

Then :

(i) *there exists a constant C such that*

$$\|\rho_\varepsilon\| + F_\varepsilon(\rho_\varepsilon) \leq C < +\infty \quad \text{for every } \varepsilon > 0.$$

(ii) *any weak* cluster point of (ρ_ε) is of the form $\rho = u \mathcal{L}^d \llcorner \Omega$ with $u \in L^1(\Omega)$.*

Remark 2.6. The proposition above implies that sequences of point configurations S_ε with equi-bounded energies admit a finite limiting intensity factor $\kappa := \limsup_\varepsilon \varepsilon^d \sharp(S_\varepsilon)$ while, by the assertion (ii), S_ε is not allowed to concentrate anywhere as $\varepsilon \rightarrow 0$.

Proof. Suppose that $\|\rho_\varepsilon\|$ has no upper bound. Then the uniform energy upper bound (2.7) implies that

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(\rho_\varepsilon)}{\|\rho_\varepsilon\|} \leq \sup_{\bar{\Omega}} |U|,$$

while by (2.5) the left hand member of the previous inequality is infinite. So there is a contradiction and we can conclude that (ρ_ε) is bounded. Then it follows from (2.7) that $F_\varepsilon(\rho_\varepsilon)$ is bounded as well, whence the assertion (i).

Let us now prove the assertion (ii); we know that the sequence (ρ_ε) is bounded and therefore admits weak* cluster points. Given such a cluster point, we can assume, without loss of generality, that $\rho_\varepsilon \xrightarrow{*} \rho$ in $\mathcal{M}_+(\bar{\Omega})$. Let us introduce for every $t > 0$ the set

$$E_t := \left\{ x \in \bar{\Omega} : \liminf_{r \rightarrow 0} \frac{\rho(B(x, r))}{\omega_d r^d} > t \right\}.$$

Thanks to the upper semi-continuity of the map $x \rightarrow \rho(B(x, r))$, we infer that E_t is a Borel subset of $\bar{\Omega}$. We are going to prove that

$$(2.8) \quad \lim_{t \rightarrow +\infty} \rho(E_t) = 0.$$

To that aim, we consider the family of closed balls in R^d defined by

$$\mathcal{F}_t := \bigcup \left\{ \overline{B(x, r)} : x \in E_t, r < r_x, \rho(\partial B(x, r)) = 0 \right\},$$

where $r_x > 0$ is chosen so that $\rho(B(x, r)) > t \omega_d r^d$ for every $r < r_x$. Since \mathcal{F}_t determines a fine covering of the bounded Borel set E_t , we may invoke the Vitali-Besicovitch covering theorem (see [1, Thm 2.19]) which provides the existence a countable subfamily (B_n) such that

$$(2.9) \quad \rho(B_n) > t \mathcal{L}^d(B_n) \quad \forall n \quad , \quad \rho(E_t \setminus \cup_n B_n) = 0.$$

Next we associate with the weak* convergent sequence (ρ_ε) , two set functions defined on Borel subsets $A \subset \bar{\Omega}$:

$$\eta_\varepsilon(A) := F_\varepsilon(\rho_\varepsilon \llcorner A) \quad , \quad \eta(A) := \liminf_{\varepsilon \rightarrow 0} \eta_\varepsilon(A).$$

We can readily check that η_ε and η are monotone with respect to the inclusion while $\eta_\varepsilon(\bar{\Omega}) = F_\varepsilon(\rho_\varepsilon)$ implies that $\eta(\bar{\Omega}) \leq \beta < +\infty$. Moreover, by the first inequality in (2.1), η_ε is super-additive on disjoint Borel subsets. Obviously this holds true also for the set function η . By

applying this property to the sequence of disjoint balls B_n , we get the upperbound:

$$(2.10) \quad \sum_n \eta(B_n) \leq C .$$

On the other hand, thanks to the coercivity inequality (2.6) that we apply with $\bar{\Omega} = B_n$, we obtain:

$$(2.11) \quad \eta_\varepsilon(B_n) = F_\varepsilon(\rho_\varepsilon \llcorner B_n) \geq \ell_-(\delta) \rho_\varepsilon(B_n) \left(\frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} - 1 \right)_+ ,$$

where $\beta_\varepsilon(B_n) := \varepsilon^d m_\delta \left(\frac{B_n}{\varepsilon} \right)$. Since $\rho(\partial B_n) = 0$ by construction and thanks to (2.3) and (2.9), we infer that

$$\lim_{\varepsilon \rightarrow 0} \frac{\rho_\varepsilon(B_n)}{\beta_\varepsilon(B_n)} = \frac{\rho(B_n)}{\delta^{-d} \mathcal{L}^d(\mathcal{B}_n)} \geq t \delta^d .$$

Therefore, passing to the limit $\varepsilon \rightarrow 0$ in (2.11), we deduce that:

$$(2.12) \quad \eta(B_n) \geq \ell_-(\delta) \rho(B_n) (t \delta^d - 1)_+ .$$

All in all, after collecting the second equality of (2.9), (2.10) and (2.12), we are led to:

$$\rho(E_t) \leq \sum_n \rho(B_n) \leq \frac{C}{\ell_-(\delta)(t \delta^d - 1)_+} .$$

Our claim (2.8) follows by sending $t \rightarrow +\infty$. The absolute continuity property $\rho \ll \mathcal{L}^d$ stated in the assertion (ii) is a consequence of the *Besicovitch differentiation theorem* [1, theorem 2.22]), which states that the singular part ρ_s in the Lebesgue-Nikodym decomposition of ρ with respect to the Lebesgue measure coincides with $\rho \llcorner E_\infty$ being $E_\infty = \bigcap_{t>0} E_t$. In our case $\|\rho_s\| = \rho(E_\infty) = 0$ due to (2.8). \square

2.3. Upper-bound of energies. In the same way as in [17], we will be using an upper bound of $\xi_\ell(S)$ when S is arranged on a d -dimensional periodic Bravais lattice \mathcal{G} ¹. To such a lattice we associate the ℓ -*Epstein zeta function* defined for every $r > 0$ by:

$$(2.13) \quad \Lambda_{\ell, \mathcal{G}}(r) := \sum_{x \in \mathcal{G} \setminus \{0\}} \ell(r|x|) .$$

In the case where the cartesian lattice $\mathcal{G} = \mathbb{Z}^d$ is used, we will write simply $\Lambda_\ell(r)$. The finiteness of this function for large r , under the condition (H3), turns out to be crucial for deriving an uniform upper bound for the scaled energy F_ε given in (1.8).

¹i.e. of the form $\mathcal{G} = F \mathbb{Z}^d$ for some invertible matrix $F \in \mathbb{R}^{d \times d}$

Lemma 2.7. *Under (H1) – (H3), there exists $C_{\mathcal{G}} > 0$ such that*

$$(2.14) \quad \Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \left(\ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt \right) \quad \forall r \geq r_0 \max\{1, a_{\mathcal{G}}^{-1}\},$$

where $a_{\mathcal{G}} := \min\{|y| : y \in \mathcal{G} \setminus \{0\}\}$.

Proof. To simplify, we chose the lattice \mathcal{G} so that $a_{\mathcal{G}} = 1$. Up to substituting ℓ with ℓ_+ which satisfies $\ell_+ \geq \ell$ and $\ell_+ = \ell$ on $[r_0, +\infty)$, we may also assume that ℓ is non-increasing on \mathbb{R}_+ . Accordingly, for any $s \in [0, \ell(0_+))$, the set of values $\{\ell > s\}$ forms a non-empty interval $[0, \ell^{-1}(s))$. The pseudo inverse $\ell^{-1}(s)$ is the supremum of all $t \geq 0$ such that $\ell(t) > s$, and it is a monotone non-increasing function on $[0, +\infty)$. Therefore, we have the following equivalence:

$$\ell^{-1}(s) > t \iff \ell(t) > s.$$

By applying the layer cake formula to the counting measure on \mathcal{G} , we get

$$\Lambda_{\ell, \mathcal{G}}(r) = \int_0^{\infty} N_r(s) ds,$$

where the integer function $N_r(s) := \#\{\{x \in \mathcal{G} \setminus \{0\} : \ell(r|x|) > s\}$ satisfies $N_r(s) = 0$ if $r \geq r_0$ and $s \geq \ell(r_0)$ (we assumed that $a_{\mathcal{G}} = 1$). On the other hand, for any periodic Bravais lattice $\mathcal{G} \subset \mathbb{R}^d$, there exists a constant $C_{\mathcal{G}} > 0$ such that

$$\#\{B_r \cap \mathcal{G}\} \leq C_{\mathcal{G}} r^d, \quad \forall r > 0.$$

This implies the inequality:

$$N_r(s) = \#\{\{x \in \mathcal{G} \setminus \{0\} : r|x| < \ell^{-1}(s)\}\} \leq \frac{C_{\mathcal{G}}}{r^d} (\ell^{-1}(s))^d.$$

Therefore, for every $r \geq r_0$, we are led to:

$$\Lambda_{\ell, \mathcal{G}}(r) \leq \frac{C_{\mathcal{G}}}{r^d} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds.$$

Then, after noticing that $\{s \in [0, \ell(r_0)] : \ell^{-1}(s) > t\} = [0, \ell(r_0) \wedge \ell(t)]$ holds for any $t \geq 0$, we obtain the desired inequality by applying once again the layer cake formula:

$$\begin{aligned} \int_0^{\ell(r_0)} (\ell^{-1}(s))^d ds &= d \int_0^{\infty} (\ell(r_0) \wedge \ell(t)) t^{d-1} dt \\ &= \ell(r_0) r_0^d + d \int_{r_0}^{+\infty} t^{d-1} \ell(t) dt. \end{aligned}$$

□

Remark 2.8. In view of Lemma 2.7, the ℓ - Epstein zeta function $\Lambda_{\ell, \mathcal{G}}(r)$ vanishes at infinity. However the behavior in $O(r^{-d})$ as $r \rightarrow \infty$ suggested by (2.14) is not optimal as we can see in the case of a Riesz potential $\ell(r) = r^{-s}$ with $s > d$, where $\Lambda_{\ell, \mathcal{G}}(r) = C r^{-s}$. On the other

hand, it is noteworthy that $\Lambda_{\ell, \mathcal{G}}$ is not continuous in general. A very simple example to see this is given by the step function $\ell = \frac{1}{2} \mathbf{1}_{[0,1]}$ which satisfies (H1) – (H3). For $d = 1$ and the lattice $\mathcal{G} = \mathbb{Z}$, we find that

$$\Lambda_{\ell}(r) = \#\{n \in \mathbb{N} : 0 < nr < 1\} = [r^{-1}] ,$$

where $[\cdot]$ denotes the integer part.

Next, by applying Lemma 2.7 in the case of the Cartesian lattice $\mathcal{G} = \mathbb{Z}^d$, we derive a fundamental upper bound for the short-range interaction energy.

Lemma 2.9. *Let $r > 0$ and S be a finite subset of the lattice $r\mathbb{Z}^d$. Then*

$$(2.15) \quad \xi_{\ell}(S) \leq \#(S) \Lambda_{\ell}(r)$$

As a consequence, for every $a > 0$, there exists $S_{\varepsilon} \subset \bar{\Omega}$ such that $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$ satisfies

$$(2.16) \quad \rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega \quad \text{and} \quad \limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|$$

where the right hand side upper bound is finite whenever $0 \leq a \leq r_0^{-d}$.

Proof. Let $S = \{r x_i : 1 \leq i \leq N\}$ where $N = \#(S)$ and $x_i \in \mathbb{Z}^d$. Noticing that, for every i , the set $\{x_i - x_j : j \neq i\}$ consists of $N - 1$ distinct elements of $\mathbb{Z}^d \setminus \{0\}$, we infer that

$$\sum_{j \neq i} \ell(r|x_i - x_j|) \leq \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \ell(r|z|) = \Lambda_{\ell}(r),$$

hence the desired inequality (2.15) by summing with respect to i .

Taking now $\rho_{\varepsilon} = \rho_{S_{\varepsilon}}^{\varepsilon}$ where $S_{\varepsilon} = \Omega \cap (r_{\varepsilon}\mathbb{Z})^d$ and $r_{\varepsilon} = \varepsilon a^{-\frac{1}{d}}$, we obtain a sequence such that $\rho_{\varepsilon} \xrightarrow{*} a \mathcal{L}^d \llcorner \Omega$ as $\varepsilon \rightarrow 0$. Indeed, by the periodicity of the Euclidean lattice, ρ_{ε} converges to a uniform density on Ω while its total mass $\|\rho_{\varepsilon}\| = \varepsilon^d \#(S_{\varepsilon}) \sim \varepsilon^d (r_{\varepsilon}^{-d} |\Omega|)$ converges to $a |\Omega|$ as $\varepsilon \rightarrow 0$. Eventually, by applying (2.15), we get $F_{\varepsilon}(\rho_{\varepsilon}) = \varepsilon^d \xi_{\ell}(\frac{S_{\varepsilon}}{\varepsilon}) \leq \varepsilon^d N_{\varepsilon} \Lambda_{\ell}(a^{-1/d})$, whence:

$$\limsup_{\varepsilon} F_{\varepsilon}(\rho_{\varepsilon}) \leq a \Lambda_{\ell}(a^{-\frac{1}{d}}) |\Omega|.$$

The finiteness of $\Lambda_{\ell}(a^{-\frac{1}{d}})$ for $a \in [0, r_0^{-d}]$ follows from Lemma 2.7. \square

Remark 2.10. The upper bound (2.15) and (2.16) obtained by choosing $\mathcal{G} = \mathbb{Z}^d$ as the reference lattice are in general not optimal since the Epstein-zeta function of another lattice could provide better ones. Obviously the inequality (2.15) holds true after replacing Λ_{ℓ} with the Epstein function $\Lambda_{\ell, \mathcal{G}}$ of any Bravais lattice $\mathcal{G} = F\mathbb{Z}^d$, while for the validity of (2.15), we need to add the normalization condition $|\det(F)| = 1$

(thereby fixing the volume of the so called *fundamental domain* of \mathcal{G}). The existence and the determination of an optimal lattice \mathcal{G} for the following minimization problem:

$$\inf\{\Lambda_{\ell,\mathcal{G}}(r) : \mathcal{G} = F Z^d, |\det(F)| = 1\}$$

touches on a very hard and famous problem related to crystallisation conjectures (see [17, 4]). Note that, for a general cost ℓ , the answer to this problem will depend of the value r (thus of the local density $a = \frac{d\rho}{dx}$ of the limiting measure ρ).

3. THE Γ -CONVERGENCE RESULT.

3.1. A quick overlook. The notion of Γ -convergence is popular in the community of calculus of variations and very much used in the analysis of sharp-interface models, dimension reduction for problems in mechanics, optimal design and homogenization. As pointed out in the introduction, this tool is also perfectly suited to justify a mean-field approach for large particle systems subject to a minimum energy criterion (see [21]). For the convenience of the reader, let us give here some basic definitions and main properties. For further details, we refer to the monographs [2, 11, 7].

Let (E, τ) be a metrizable topological space and consider a sequence of functionals $F_n : E \rightarrow (-\infty, +\infty]$. Then the lower Γ -limit F_- and the upper Γ -limit F_+ of F_n are defined by:

$$F_-(u) := \inf_{u_n \rightarrow u} \liminf_{n \rightarrow \infty} F_n(u_n) \quad , \quad F_+(u) := \inf_{u_n \rightarrow u} \limsup_{n \rightarrow \infty} F_n(u_n).$$

Both are τ -lower semicontinuous (see [2]), whereas in general it holds $F_- \leq F_+$. In practice it is useful to check that these functionals are proper i.e. that they range into $\mathbb{R} \cup \{+\infty\}$ being not identically $+\infty$. If F_n admits a lower bound independent of n , this amounts to checking that the existence of $u_0 \in E$ such that

$$(3.1) \quad F_+(u_0) = \inf_{u_n \rightarrow u_0} \limsup_{n \rightarrow \infty} F_n(u_n) < +\infty.$$

We say that F_n Γ -converges to F (denoted $F_n \xrightarrow{\Gamma} F$) if $F = F_- = F_+$ or equivalently if the two following conditions are fulfilled:

- a) (*lowerbound*) For any sequence u_n converging to u , we have the inequality

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F(u);$$

- b) (*recovering sequence*) For every $u \in X$ such that $F(u) < +\infty$, there exists (u_n) such that

$$u_n \rightarrow u \quad \text{and} \quad F_n(u_n) \rightarrow F(u).$$

This is the case in particular if $F_n = F$ does not depend of n ; then $F_n \xrightarrow{\Gamma} \text{cl}(F)$ where $\text{cl}(F)$ denotes the τ -lower semicontinuous envelope of F . Among all properties of Γ -convergence, we give some which will be used in this paper.

Proposition 3.1. *Let $F_n : E \rightarrow (-\infty, +\infty]$ and assume (3.1). Then:*

- (i) $F_n \xrightarrow{\Gamma} F \iff \text{cl}(F_n) \xrightarrow{\Gamma} F$;
- (ii) (Kuratowski compactness Theorem) *If (E, τ) is a second countable topological space (for instance a separable metric space), then any sequence (F_n) admits a Γ -convergent subsequence;*
- (iii) (convergence of infima) *Suppose that $F_n \xrightarrow{\Gamma} F$ and that the following equi-coercivity property holds:*

$$\sup_n F_n(u_n) < +\infty \implies \{u_n\} \text{ is } \tau\text{-relatively compact.}$$

Then $\lim_{n \rightarrow \infty} \inf_X F_n = \min_X F$ and the minimum set for F coincides with the cluster points of all sequences (u_n) such that $F_n(u_n) - \inf F_n \rightarrow 0$;

- (iv) (stability) $F_n \xrightarrow{\Gamma} F \implies F_n + G \xrightarrow{\Gamma} F + G$ for every continuous perturbation function $G : E \rightarrow \mathbb{R}$.

Remark 3.2. The continuity requirement for G in the assertion (iv) is often too restrictive. Actually the same conclusion holds under the following milder condition:

$$(3.2) \quad \begin{cases} \inf_K G > -\infty & \text{for any compact } K \subset E \\ G(u_n) \rightarrow G(u) & \text{whenever } u_n \rightarrow u \text{ and } F(u) < +\infty \end{cases}$$

For the convenience of the reader, a brief proof of the sufficiency of this condition is given below.

Proof. To check condition a), we consider a sequence (u_n) such that $u_n \rightarrow u$. Without loss of generality, assume that $F_n(u_n) + G(u_n) \leq C$ for a suitable constant C . Since $G(u_n)$ is lower bounded (take K to be $\{u_n, n \in \mathbb{N}\} \cup \{u\}$), we infer that $F_n(u_n) \leq C'$ for another constant C' . By the Γ -convergence $F_n \xrightarrow{\Gamma} F$, it follows that

$$F(u) \leq \liminf_n F_n(u_n) < +\infty.$$

Therefore $G(u_n) \rightarrow G(u)$ and $\liminf_n (F_n + G)(u_n) \geq (F + G)(u)$.

For checking condition b), we may restrict to elements $u \in E$ such that $(F + G)(u) < +\infty$. Then $F(u) < +\infty$ and any recovering sequence $u_n \rightarrow u$ such that $F_n(u_n) \rightarrow F(u)$ will satisfy $G(u_n) \rightarrow G(u)$. \square

3.2. The main result. In our context, the Γ -convergence issue applies to the sequence (F_ε) defined in (1.8) and to the ambient topological space $\mathcal{M}_+(\overline{\Omega})$ embedded with the weak* topology (tight convergence). Thanks to the equi-coercivity property established in Proposition 2.5, there is no loss of generality in working in a fixed closed ball of $\mathcal{M}_+(\overline{\Omega})$ which is metrizable and compact. Therefore all the properties mentioned in the former subsection are applicable (after substituting the index $n \rightarrow \infty$ with the continuous parameter $\varepsilon \rightarrow 0$).

Theorem 3.3. *Let ℓ satisfy the standing assumptions (H1) – (H3) and let $F_\varepsilon : \mathcal{M}_+(\overline{\Omega}) \mapsto [0, +\infty]$ be given by (1.8). Then $F_\varepsilon \xrightarrow{\Gamma} F$ (for the weak* topology) where*

$$F(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

and $f_\ell : \mathbb{R}_+ \rightarrow [0, +\infty]$ is convex, l.s.c. and satisfies

$$(3.3) \quad f_\ell(0) = f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} > 0.$$

The proof of Theorem 3.3 is postponed to Section 5 while the property (3.3) is established in the next Sub-section. Let us now consider the Fenchel conjugate of f_ℓ (implicitly extended by $+\infty$ on $(-\infty, 0)$) given by

$$(3.4) \quad f_\ell^*(\lambda) := \sup \{ \lambda t - f_\ell(t) : t \in \mathbb{R}_+ \}$$

From (3.3), one can check that the supremum in (3.4) is actually a maximum (which is attained at $t = 0$ if $\lambda \leq 0$). Therefore, f_ℓ^* is convex, continuous and vanishes on $(-\infty, 0]$. As a consequence it admits left and right derivatives $(f_\ell^*)'(\lambda_-) \leq (f_\ell^*)'(\lambda_+)$ for any λ so that the subdifferential $\partial f_\ell^*(\lambda) = [(f_\ell^*)'(\lambda_-), (f_\ell^*)'(\lambda_+)]$ is non empty.

Corollary 3.4. *Let $U \in \mathcal{C}(\overline{\Omega})$ be an external potential. Then, for every $\varepsilon > 0$, there exists a finite set $S_\varepsilon \subset \overline{\Omega}$ minimizing*

$$\mathcal{I}_\ell^{(\varepsilon)}(U) := \inf_{S \subset \overline{\Omega}} \left\{ \sum_{(x,y) \in S^2 \setminus \Delta} \ell \left(\frac{|x-y|}{\varepsilon} \right) + \sum_{x \in S} U(x) \right\}.$$

Moreover $\sup_\varepsilon \varepsilon^d \sharp(S_\varepsilon) < +\infty$ and $\lim_{\varepsilon \rightarrow 0} \varepsilon^d \mathcal{I}_\ell^{(\varepsilon)}(U) = \mathcal{I}_\ell(U)$, where

$$(3.5) \quad \mathcal{I}_\ell(U) := \min_{u \in L^1(\Omega)} \left\{ \int_{\Omega} (f_\ell(u) + uU) dx \right\} = - \int_{\Omega} f_\ell^*(-U) dx.$$

Furthermore any weak* cluster point of $\rho_{S_\varepsilon}^\varepsilon$ belongs to the minimum set of (3.5) given by

$$(3.6) \quad \mathcal{S}_\ell := \{ u \mathcal{L}^d \llcorner \Omega : u(x) \in \partial f_\ell^*(-U(x)) \text{ a.e. } x \in \Omega \}.$$

Remark 3.5 (*Non-uniqueness*). In general f_ℓ is not strictly convex and ∂f_ℓ^* can be multi-valued (see for instance the hard spheres case depicted in Subsection 3.4 or the example of the step function ℓ given in (3.19) where f_ℓ is piecewise affine). Note that, since f_ℓ^* vanishes on \mathbb{R}_- , it holds $\mathcal{I}_\ell(U) = 0$ for every non-negative potential U .

Remark 3.6. A natural variant of $\mathcal{I}_\ell^{(\varepsilon)}(U)$ consists in prescribing the total number N_ε of particles to satisfy $N_\varepsilon \sim \kappa \varepsilon^{-d}$ for some given intensity factor $\kappa \in (0, +\infty)$. Accordingly, Corollary 3.4 can be restated by adding a total mass constraint in the limit problem, that is with $I_\ell(U)$ in (3.5) replaced by

$$\mathcal{I}_{\ell,\kappa}(U) := \inf \left\{ \int_{\Omega} (f_\ell(u) + uU) dx : u \in L^1(\Omega; \mathbb{R}_+) , \int_{\Omega} u dx = \kappa \right\}.$$

The associated minimum $\mathcal{S}_{\ell,\kappa}$ can be determined by selecting a suitable Lagrange multiplier depending implicitly on κ ; as a consequence, an explicit form for $\mathcal{S}_{\ell,\kappa}$ of the kind (3.6) is not available.

Remark 3.7 (*Clustering*). For $\varepsilon > 0$ fixed, the lower semicontinuity property of F_ε requires that $\ell(0+) = +\infty$. Otherwise, if $\ell(0+) < +\infty$, a sequence of subsets $S_n \subset \bar{\Omega}$ such that $\sup_n F_\varepsilon(\rho_{S_n}^\varepsilon) < +\infty$ (thus retaining a finite number of points) can collapse into several clusters while retaining finite energy. In this case, the relaxed functional $\text{cl}(F_\varepsilon)$ can be obtained directly by extending its domain to ε -empirical measures associated with multisets (instead of sets) and by extending the definition (1.6) accordingly taking into account the multiplicity of each cluster of particles. Note however that considering $\text{cl}(F_\varepsilon)$ instead of F_ε will not change the mean-field energy F given in Theorem 3.3 in virtue of the assertion i) of Proposition 3.1.

3.3. Characterization and properties of f_ℓ . The convex integrand f_ℓ will be characterized indirectly through its Fenchel conjugate. For every $\lambda \in \mathbb{R}$ and any Borel subset $B \subset \mathbb{R}^d$, we define:

$$(3.7) \quad \Gamma_\ell(\lambda, B) := \sup \{ \lambda \#(S) - \xi_\ell(S) : S \text{ finite } \subset B \}.$$

The key properties of this bivariate function are summarized in the two following lemmas. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the convex, continuous function defined by:

$$(3.8) \quad \varphi(t) = \begin{cases} \frac{(1+t)^2}{4} & \text{if } t \geq 1 \\ t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t < 0 \end{cases}.$$

A straightforward computation shows that its Fenchel conjugate is given by:

$$\varphi^*(\zeta) = \zeta(\zeta - 1)_+ \quad \text{if } \zeta \geq 0 \quad , \quad \varphi^*(\zeta) = +\infty \quad \text{if } \zeta < 0.$$

Lemma 3.8. *Assume that $\mathcal{L}^d(B) < +\infty$. Then the map $\lambda \rightarrow \Gamma_\ell(\lambda, B)$ is convex, continuous, vanishes on \mathbb{R}_- and satisfies:*

$$(3.9) \quad 0 \leq \Gamma_\ell(\lambda, B) \leq m_\delta(B) \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

for every $\lambda \in \mathbb{R}$, being $\delta > 0$ such that $\ell_-(\delta) \in (0, +\infty]$ and φ being defined by (3.8).

Notice that the right hand side of (3.9) is convex as a function λ . On the other hand, due to the linear behavior of φ on $[0, 1]$, we can easily infer that $\Gamma_\ell(\lambda, B) \leq \lambda m_\delta(B)$ whenever $\ell_-(\delta) = +\infty$.

Proof. As a supremum of affine functions, $\Gamma_\ell(\lambda, B)$ is convex and l.s.c. with respect to λ ; it is non-negative (follows by taking S the empty set) and vanishes if $\lambda \leq 0$. The continuity property follows classically from (3.9) which provides the finiteness of $\Gamma_\ell(\lambda, B)$ since $m_\delta(B) < +\infty$ once $\mathcal{L}^d(B) < +\infty$. It remains to prove the upper bound in (3.9) which clearly follows from (2.4). Indeed, in term of $\zeta = \frac{\#(S)}{m_\delta(B)}$, we have for every finite $S \subset B$:

$$\lambda \#(S) - \xi_\ell(S) \leq m_\delta(B) (\lambda \zeta - \ell_-(\delta) \zeta (\zeta - 1)_+).$$

Taking the supremum of the right hand member with respect to $\zeta \geq 0$ and noticing that $\zeta (\zeta - 1)_+ = \varphi^*(\zeta)$, we derive (3.9) after some easy manipulations. □

Lemma 3.9. *For every $\lambda \geq 0$, the set function $B \mapsto \Gamma_\ell(\lambda, B)$ is sub-additive on disjoint Borel subsets and translation invariant.*

Proof. Let B_1, B_2 be such that $B_1 \cap B_2 = \emptyset$ and let S be a finite subset of $B_1 \cup B_2$. Then, setting $S_i = S \cap B_i$, we get a partition $S = S_1 \cup S_2$ and by applying the super-additivity part of (2.1), we infer that:

$$\lambda \#(S) - \xi_\ell(S) \leq \sum_{i=1}^2 (\lambda \#(S_i) - \xi_\ell(S_i)) \leq \Gamma_\ell(\lambda, B_1) + \Gamma_\ell(\lambda, B_2),$$

hence the desired sub-additivity property by taking the supremum with respect to $S \subset B_1 \cup B_2$. The invariance by translation is trivial. □

In virtue of Lemma 3.9, we may now apply to $\Gamma(\lambda, \cdot)$ a classical result by Krengel [16], which ensures the existence, for every λ , of a limit for the ratio $\frac{\Gamma(\lambda, Q_k)}{k^d}$ as $k \rightarrow +\infty$ (*thermodynamical limit*). Let us define the function

$$(3.10) \quad g_\ell(\lambda) := \inf_{k \in \mathbb{N}} \frac{\Gamma(\lambda, Q_k)}{k^d}.$$

Then we have:

$$(3.11) \quad g_\ell(\lambda) = \lim_{k \rightarrow +\infty} \frac{\Gamma(\lambda, Q_k)}{k^d} = \lim_{\varepsilon \rightarrow 0} \varepsilon^d \Gamma(\lambda, Q_{1/\varepsilon}).$$

For a proof of (3.11), we refer for instance to [18]. Since the effective profile f_ℓ in Theorem 3.3 will be identified through the relation $f_\ell^* = g_\ell$ (see the last step of the proof in Section 5), we now establish some useful bounds for g_ℓ . Recalling the definition of Λ_ℓ in (2.13), we introduce the function defined on \mathbb{R} by:

$$H_\ell(t) := \begin{cases} t \Lambda_\ell(t^{-\frac{1}{d}}) & \text{if } t > 0, \\ +\infty & \text{if } t \leq 0. \end{cases}$$

From $(H_1)(H_2)$, it is straightforward that this function H_ℓ is l.s.c. and finite on $(0, r_0^{-d}]$ while $H_\ell(0_+) = H'_\ell(0_+) = 0$. Note that H_ℓ is not convex in general, even it could be discontinuous, as happens for ℓ which is the step function $\ell(r) = \frac{1}{2} \mathbf{1}_{[0,1]}$. Indeed, in this case and for $d = 1$, we get $H_\ell(t) = t [t]$ (see Remark 2.8). However H_ℓ is convex continuous on \mathbb{R}_+ in many classical cases including that of hyper singular Riesz potentials $\ell(r) = r^{-s}$ where $H_\ell(t) = C t^{1+s/d}$.

Proposition 3.10. *The function $g_\ell(\lambda)$ is convex, continuous, non-negative, vanishes on \mathbb{R}_- and, for $\lambda > 0$, satisfies the inequalities:*

$$(3.12) \quad H_\ell^*(\lambda) \leq g_\ell(\lambda) \leq \delta^{-d} \ell_-(\delta) \varphi\left(\frac{\lambda}{\ell_-(\delta)}\right),$$

being φ given by (3.8) and $\delta > 0$ chosen such that $\ell_-(\delta) > 0$.²

Proof. In view of (3.11) and of the convexity of $\Gamma(\cdot, Q_k)$, the function g_λ is convex, non-negative and vanishes for $\lambda \leq 0$ as a pointwise limit as $k \rightarrow \infty$ of the sequence of functions $k^{-d} \Gamma(\cdot, Q_k)$.

To prove the right hand inequality in (3.12), it is enough to apply (3.9) with $B = Q_k$ and, after dividing by k^d , pass to the limit taking into account that, in virtue of (2.3), $m_\delta(Q_k) \sim (\frac{k}{\delta})^d$ as $k \rightarrow \infty$. Since the majorant is a convex, continuous function of λ , we infer that g_ℓ is continuous on \mathbb{R} as well.

Eventually, let us apply, for every $t > 0$ and $k \in \mathbb{N}_*$, the fundamental upper bound (2.15) to the finite subset $S_{t,k} := t^{-\frac{1}{d}} Z^d \cap Q_k$. Then, recalling (3.7), we have:

$$\Gamma_\ell(\lambda, Q_k) \geq \lambda \#(S_{t,k}) - \xi_\ell(S_{t,k}) \geq \#(S_{t,k}) \left(\lambda - \Lambda_\ell(t^{-\frac{1}{d}}) \right).$$

Since $\#(S_{t,k}) \sim t k^d$ as $k \rightarrow \infty$ and in virtue of (3.11), we are led to:

$$g_\ell(\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(\lambda, Q_k)}{k^d} \geq t(\lambda - \Lambda_\ell(t^{-\frac{1}{d}})) = \lambda t - H_\ell(t).$$

The left hand side inequality in (3.12) follows by taking the supremum with respect to t . \square

²The second inequality becomes $g_\ell(\lambda) \leq \delta^{-d} \lambda$ in the case where $\ell_-(\delta) = +\infty$.

Corollary 3.11. *The integrand $f_\ell := g_\ell^*$ is convex, l.s.c. and satisfies $f_\ell(0) = 0$ while $f_\ell(t) = +\infty$ for $t < 0$. Moreover, for every $t > 0$ and $\delta > 0$, we have the inequalities:*

$$(3.13) \quad \ell_-(\delta) t (t\delta^d - 1)_+ \leq f_\ell(t) \leq H_\ell^{**}(t).$$

Accordingly, f_ℓ is finite on $[0, r_0^{-d}]$, monotone non-decreasing on $[0, +\infty)$ and satisfies:

$$(3.14) \quad f'_\ell(0_+) = 0 \quad , \quad \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \geq \sup_{\delta > 0} \ell_-(\delta)\delta^d > 0$$

Proof. Passing carefully to Fenchel conjugates in inequalities (3.12), we are led to (3.13) from which the other statements follow directly. In particular, as $f'_\ell(0_+) = 0$, we infer that the convex function f_ℓ is monotone non-decreasing on \mathbb{R}_+ . \square

Remark 3.12 (*growth conditions*). The inequality in (3.14) confirms that f_ℓ grows at least quadratically at infinity, as announced in the introduction (see Theorem 3.3). More specifically, we can highlight two subcases for a cost ℓ satisfying $(H1) - (H3)$.

- a) $k_\ell := \sup_{\delta > 0} \ell_-(\delta)\delta^d = +\infty$. Then $\liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} = +\infty$ and f_ℓ has a super quadratic growth. Note that this conclusion is consistent with the case $\ell(r) = r^{-s}$ for $s > d$ (see the next subsection).
- b) $\int_0^\infty \ell_+(t)t^{d-1} dt < +\infty$. In this case $k_\ell < +\infty$ and, thanks to (3.13) and to the estimate given in Lemma 2.7 (that we can apply to ℓ_+ with $r_0 = 0$), we obtain the lower and upper bounds:

$$0 < k_\ell \leq \liminf_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq \limsup_{t \rightarrow +\infty} \frac{f_\ell(t)}{t^2} \leq C \int_0^\infty \ell_+(t)t^{d-1} dt.$$

It follows that, under the integrability condition $\int_{\mathbb{R}^d} \ell_+(|x|) dx < +\infty$, f_ℓ enjoys a quadratic growth from above and from below.

3.4. Examples.

3.4.1. *The hard spheres model.* The hard spheres potential is given by

$$\ell(r) = \begin{cases} +\infty & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}$$

The computation of g_ℓ through (3.10) and (3.11) leads to a linear function on \mathbb{R}_+ namely $g_\ell(\lambda) = \gamma_d \lambda$, where γ_d denotes the densest spheres packing volume fraction in \mathbb{R}^d . This famous universal constant can be defined as

$$(3.15) \quad \gamma_d := \inf_{k \in \mathbb{N}^*} \frac{S(Q_k)}{k^d} = \lim_{k \rightarrow \infty} \frac{S(Q_k)}{k^d},$$

where, for any Borel set $A \subset \mathbb{R}^d$, $S(A)$ denotes the maximal number of points in A with mutual distance larger or equal to 1. The mean-field energy density f_ℓ given by Theorem 3.3 is therefore the indicator function of the interval $[0, \gamma_d]$

$$f_\ell(t) = 0 \quad \text{if } t \leq \gamma_d \quad , \quad f_\ell(t) = +\infty \quad \text{otherwise .}$$

Furthermore, for every continuous external potential $U \in \mathcal{C}(\bar{\Omega})$, we recover from Corollary 3.4 the convergence:

$$\min_{S \in \mathcal{F}_\varepsilon(\bar{\Omega})} \left\{ \varepsilon^d \sum_{z \in S} U(z) \right\} \rightarrow \gamma_d \int_{\bar{\Omega}} U(z) dz ,$$

where $\mathcal{F}_\varepsilon(\bar{\Omega})$ is the family of finite subsets $S \subset \bar{\Omega}$ satisfying $|x - y| \geq \varepsilon$ for all $(x, y) \in S^2 \setminus \Delta$.

Remark 3.13. A variant of the previous result was obtained recently in [3] in the case where the total number of particles N_ε is prescribed to satisfy $N_\varepsilon \varepsilon^d \rightarrow \kappa$ as $\varepsilon \rightarrow 0$ where $\kappa > 0$ is a given real parameter. With our notations this condition amounts to restrict the Γ -limit F to measures ρ such that $\int_{\Omega} u dx = \kappa$. Since the domain of F consists of density measures $\rho = u \mathcal{L}^d \llcorner \Omega$ such that $u \leq \gamma_d$ a.e., the latter integral condition requires that $\kappa \leq \gamma_d |\Omega|$ hence a congestion ratio $\theta := \frac{\kappa}{\gamma_d |\Omega|}$ not larger than 1. In this case and if, following the classical empirical measure representation, u is normalized to be a probability density by setting $\tilde{u} := \frac{u}{\int_{\Omega} u dx} = \frac{u}{\kappa}$, we recover a mean-field energy vanishing for $\tilde{u} \leq \frac{\gamma_d}{\kappa}$ and infinite otherwise, exactly as stated in [3, Thm 6.1]. Note that the duality technique used there could only handle cost functions ℓ taking values in the discrete set $\{0, +\infty\}$.

3.4.2. *The case of Riesz potentials.* Short range potential of Riesz type corresponds to fixing $s > d$ and taking

$$l(r) = r^{-s} \text{ on } \mathbb{R}_+^d .$$

In this case, it's easy to establish from the homogeneity of the cost ℓ that the bivariate function Γ_ℓ defined in (3.7) satisfies, for every $t \geq 0$, the following scaling law :

$$(3.16) \quad \Gamma_\ell(t\lambda, B) = t \Gamma_\ell(\lambda, t^{1/s} B) .$$

It follows from (3.16) that:

$$\frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \frac{t \Gamma_\ell(\lambda, t^{1/s} Q_k)}{k^d} = t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} .$$

Sending $k \rightarrow \infty$ and applying (3.11) two times, we get:

$$g_\ell(t\lambda) = \lim_{k \rightarrow \infty} \frac{\Gamma_\ell(t\lambda, Q_k)}{k^d} = \lim_{k \rightarrow \infty} t^{1+d/s} \frac{\Gamma_\ell(\lambda, Q_{t^{1/s} k})}{(t^{1/s} k)^d} = t^{1+d/s} g_\ell(\lambda) .$$

In virtue of the equality $f_\ell = g_\ell^*$, we deduce that:

$$(3.17) \quad f_\ell(t) = C(s, d) t^{1+s/d}$$

where $C(s, d) = f_\ell(1)$ is a universal constant. We thus recover the Γ -convergence result proved in [12, 13].

Remark 3.14. If we chose $\ell(r) = r^{-s}$ where $s < d$, then condition (H3) is violated, and the scaling defined in equation (1.8) that we used to define F_ε will result in an infinite Γ -limit. This means that $F(\rho)$ will be equal to $+\infty$ whenever ρ is not equal to zero, and $F(0)$ will be equal to zero. We can observe this when we consider a system of N_ε particles in $S_\varepsilon \subset \bar{\Omega}$ such that $\rho_\varepsilon := \rho_{S_\varepsilon}^\varepsilon$ converges weakly to ρ , and $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < +\infty$. Assuming $\rho \neq 0$, then we have $N_\varepsilon \sim \|\rho\| \varepsilon^{-d}$ as $\varepsilon \rightarrow 0$. Moreover, due to the power law property of ℓ , we can write:

$$(3.18) \quad F_\varepsilon(\rho_\varepsilon) = \varepsilon^{s+d} \xi_\ell(S_\varepsilon) \sim \frac{\|\rho\|^2}{\varepsilon^{d-s}} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2}.$$

As ℓ satisfies (1.3), the convergence result of the long range case holds (with $h_N = N^{-2}$, see [3], [20]). Therefore, based on (1.4), and given that the standard empirical measure linked to S_ε converges to $\hat{\rho} = \frac{\rho}{\|\rho\|} \in \mathcal{P}(\bar{\Omega})$, it follows that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\xi_\ell(S_\varepsilon)}{N_\varepsilon^2} \geq D_\ell(\hat{\rho}) > 0.$$

This contradicts (3.18) since $\sup_\varepsilon F_\varepsilon(\rho_\varepsilon) < \infty$. Therefore $\rho = 0$.

3.4.3. The case of finite costs. Many examples of finite costs can be considered as, for instance, ℓ being a step function with compact support. Owing to Corollary 3.11, the effective convex integrand f_ℓ has a quadratic growth on \mathbb{R}_+ . The simplest one is the penalized version of the hard spheres potential defined by:

$$(3.19) \quad \ell(r) = \begin{cases} \frac{M}{2} & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad (M \text{ positive parameter}).$$

Applying the lower bound (3.13) with $\delta = 1$, we deduce that $f_\ell \geq h$ where

$$h(t) := \frac{M}{2} t(t-1)_+.$$

In turn this lower bound is optimal for integer values of t since, as proved below, f_ℓ coincides on \mathbb{R}_+ with the piecewise affine interpolation of h given by:

$$(3.20) \quad f_\ell(t) = h(k) + (t-k)(h(k+1) - h(k)) \quad \forall t \in [k, k+1], k \in \mathbb{N}.$$

Proof. Owing to (3.11), the Fenchel conjugate of f_ℓ is given by

$$g_\ell(\lambda) = \lim_{K \rightarrow +\infty} \frac{1}{K} \sup_{S \subset [0, K]} \left\{ \lambda \#(S) - \frac{M}{2} \#(\{(x, y) \in S^2 \setminus \Delta_1\}) \right\},$$

where $\Delta_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \geq 1\}$. Let $S \subset [0, K]$ be an optimal set which we split in K disjoint pieces namely $S = \bigcup_{i=1}^K S_i$ where $S_i = S \cap [i, i + 1)$. Let us denote n_i the number of points in S_i . By pushing them to the center of the interval $[i, i + 1)$, we see that the number of pairs in $S^2 \setminus \Delta_1$ decreases to $\sum_{i=1}^K n_i(n_i - 1)$. It follows that:

$$\begin{aligned} g_\ell(\lambda) &= \lim_{K \rightarrow \infty} \sup_{n_i \in \mathbb{N}} \frac{1}{K} \left\{ \lambda \sum_{i=1}^K n_i - \frac{M}{2} \sum_{i=1}^K n_i(n_i - 1) \right\} \\ &= \sup_{n \in \mathbb{N}} \left\{ \lambda n - \frac{M}{2} n(n - 1) \right\} = (h + \chi_{\mathbb{N}})^*(\lambda), \end{aligned}$$

where $\chi_{\mathbb{N}}$ denotes the indicator function of the integers. Therefore $f_\ell = (g_\ell)^*$ is nothing else but the convexification of $h + \chi_{\mathbb{N}}$ given by the interpolation formula (3.20). \square

As demonstrated above, optimal point configurations for a constant external potential are obtained by periodically grouping a suitable number of points. Therefore, optimal sets are essentially multisets (see Remark 3.7). It is probable that a similar phenomenon occurs in higher dimensions.

The situation will vary if we consider a non-monotonic step function, such as the following:

$$\ell(r) = 1 \text{ for } r \in [0, 1], \quad \ell(r) = 4 \text{ for } r \in (1, 2), \quad \ell(r) = 0 \text{ for } r \geq 2.$$

In this case, we anticipate that optimal configurations may be periodic, but associated with a non-uniform Voronoi tessellation, consisting of patterns of different sizes, as observed in the context of optimal location problem (see [5, Sec 3.4]).

4. PROOF OF THE MAIN THEOREM

First, we check the properness property (2.16) to make sure that the upper Γ -limit of F_ε is not trivial. To do this, it is sufficient to apply Lemma 2.9 by choosing $u_0 = a \mathbf{1}_\Omega$ for $a \in [0, r_0^{-d}]$. Next, by virtue of the equi-coercivity property of F_ε proved in Proposition 2.5 and of the Kuratowski compactness theorem (see Proposition 3.1 and the introductory comment of Subsection 3.2), we can find a sequence $\varepsilon_k \rightarrow 0$ and a weak* lower semicontinuous functional $F : \mathcal{M}_+(\Omega) \rightarrow [0, +\infty]$ such that $F_{\varepsilon_k} \xrightarrow{\Gamma} F$ as $k \rightarrow \infty$. Note that the limit F may a priori depend on the chosen sequence $\varepsilon_k \rightarrow 0$. Accordingly, we will complete the proof of Theorem 3.3 in two steps which are outlined below:

Step 1: we show that F is a local functional of the form

$$(4.1) \quad F(\rho) := \begin{cases} \int_\Omega j(x, u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}$$

where $j : \Omega \times \mathbb{R} \rightarrow [0, +\infty]$ is a suitable convex normal integrand such that $j(\cdot, 0) = 0$ a.e. in Ω . As a consequence F is convex, weak* l.s.c. and coincides with its Fenchel biconjugate, i.e. :

$$F(\rho) = F^{**}(\rho) = \sup_{v \in \mathcal{C}(\overline{\Omega})} \left\{ \int v d\rho - F^*(v) \right\}.$$

Step 2: we identify the Fenchel conjugate F^* in terms of the convex function g_ℓ defined in(3.10), namely:

$$F^*(v) = \int_{\Omega} g_\ell(v) dx \quad , \quad \text{for every } v \in \mathcal{C}(\overline{\Omega}).$$

It follows that the limit functional F *does not depend on the sequence* (ε_k) . Also, since (3.14) the convex function $f_\ell = g_\ell^*$ has a superlinear growth at infinity, by applying a classical result on convex functionals on measures (see for instance [6]), we obtain the equalities $F = F^{**} = F_\ell$ where:

$$F_\ell(\rho) := \begin{cases} \int_{\Omega} f_\ell(u) dx & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

This will conclude the proof of the Γ -convergence of the whole sequence (F_ε) as stated in Theorem 3.3.

Proof of Step 1. Let $\rho \in \mathcal{M}_+(\overline{\Omega})$ be such that $F(\rho) < +\infty$. Then, there exists a recovering sequence $\rho_k \xrightarrow{*} \rho$ such that $\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = F(\rho) < +\infty$. By the assertion ii) of Proposition 2.5, we infer that ρ is an absolutely continuous measure. Accordingly, there exists a functional $J : L^1(\Omega) \rightarrow [0, +\infty]$ such that

$$F(\rho) := \begin{cases} J(u) & \text{if } \rho = u \mathcal{L}^d \llcorner \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

The following result will be crucial for deriving the integral representation and the convexity of J . Its delicate proof is postponed to the end of this section.

Lemma 4.1. *The functional $J : L^1(\Omega) \rightarrow [0, +\infty]$ defined above satisfies the following:*

- (i) J is weakly lower semicontinuous and satisfies $J(0) = 0$;
- (ii) The domain of J is a subset of $L^1(\Omega; \mathbb{R}_+)$ and $J(u \mathbf{1}_A) \leq J(u)$ holds for every $u \in L^1(\Omega, \mathbb{R}_+)$ and every Borel subset $A \subset \Omega$;
- (iii) It holds $J(u + v) = J(u) + J(v)$ whenever $uv = 0$ is satisfied almost everywhere in Ω .

In view of the assertions (i) and (iii) of Lemma 4.1 and since the Lebesgue measure on Ω is atomless, we may apply a classical integral representation (see for instance Hiai and Umegaki [14, 15] or the monograph [8]) according to which there exists a suitable *convex* normal

integrand j such that (4.1) holds. Moreover, as $J \geq 0$ and $J(0) = 0$, we have $j(\cdot, 0) = 0$ a.e. while, due to the assertion (ii), the integrand j satisfies $j(x, t) = +\infty$ if $t < 0$. \square

Proof of Step 2. From Step 1 and by a classical result on integral functionals (see for instance [6]), the Fenchel conjugate of F is given for every $v \in \mathcal{C}(\overline{\Omega})$ by:

$$F^*(v) = \sup_{u \in L^1(\Omega)} \left\{ \int_{\Omega} v u \, dx - \int_{\Omega} j(x, u(x)) \, dx \right\} = \int_{\Omega} j^*(x, v(x)) \, dx.$$

Obviously we may extend this equality to all functions $v \in L^\infty(\Omega)$. Noticing that $j^*(x, 0) = -\inf j(x, \cdot) = -j(x, 0) = 0$, we observe that, for every $\lambda \in \mathbb{R}$ and for every hypercube $Q(x_0, a) \subset \Omega$, we have

$$F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = \int_{Q(x_0, a)} j^*(x, \lambda) \, dx.$$

Next we claim that, for any such an hypercube $Q(x_0, a) \subset \Omega$, the following holds:

$$(4.2) \quad F^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda)$$

Suppose that this claim is true. Then, by considering Lebesgue points of $j^*(\cdot, \lambda)$ for λ in a dense countable subset D of \mathbb{R} , we can find a Lebesgue negligible subset $N \subset \Omega$ such that $j^*(x, \lambda) = g_\ell(\lambda)$ for all $(x, \lambda) \in (\Omega \setminus N) \times D$. Thanks to the continuity of g_ℓ proved in Proposition 3.10 and to the convexity of $j^*(x, \cdot)$, the latter equality can be then extended to all $(x, \lambda) \in (\Omega \setminus N) \times \mathbb{R}$, so that we have $F^*(v) = \int_{\Omega} g_\ell(v) \, dx$ for every $v \in \mathcal{C}(\overline{\Omega})$. Hence the conclusion of Step 2 is reached and the proof of Theorem 3.3 is complete provided we can confirm (4.2).

We now focus on the proof of the equality (4.2). For $\lambda \leq 0$, this equality is trivial since $g_\ell(\lambda) = j^*(x, \lambda) = 0$. Next we observe that, for every $\lambda \geq 0$ and $Q(x_0, a) \subset \Omega$, we have:

$$\begin{aligned} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) &:= \sup \{ \lambda \rho(Q(x_0, a)) - F_{\varepsilon_k}(\rho) : \rho \in \mathcal{M}_+(\overline{\Omega}) \} \\ &= \varepsilon_k^d \sup_{S \subset \overline{\Omega}} \{ \lambda \#(S \cap Q(x_0, a)) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S \subset Q(x_0, a)} \{ \lambda \#(S) - \xi_{\varepsilon_k}(S) \} \\ &= \varepsilon_k^d \sup_{S' \subset Q(x_0, \frac{a}{\varepsilon_k})} \{ \lambda \#(S') - \xi_\ell(S'/\varepsilon_k) \} \\ &= \varepsilon_k^d \Gamma_\ell(\lambda, Q(x_0, \frac{a}{\varepsilon_k})) \end{aligned}$$

where:

- to pass from the second to the third line, we substitute any competitor $S \subset \overline{\Omega}$ with $S \cap Q(x_0, a)$ which has larger energy;

- to pass from the third line to the two last lines, we set $S = S'/\varepsilon_k$ for going back from the ε_k - scaled energy (1.7) to the ground interaction energy ξ_ℓ and ultimately recover the set function Γ_ℓ defined in (3.7).

Therefore, thanks to (3.11), we can pass to the limit $k \rightarrow +\infty$ (the position of x_0 is irrelevant) and obtain the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = a^d g_\ell(\lambda).$$

So, proving (4.2) reduces to checking the equality

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_k}^*(\lambda \mathbf{1}_{Q(x_0, a)}) = F^*(\lambda \mathbf{1}_{Q(x_0, a)}),$$

that we rewrite in the equivalent form:

$$(4.3) \quad \inf_{\rho \in \mathcal{M}_+(\overline{\Omega})} \{F_{\varepsilon_k}(\rho) - \lambda \rho(Q(x_0, a))\} \rightarrow \inf_{\rho} \{F(\rho) - \lambda \rho(Q(x_0, a))\}.$$

The left hand side infimum in (4.3) being non-positive (easily seen by taking $\rho = 0$ as a competitor), we may apply Proposition 2.5 with the choice $U = -\lambda \mathbf{1}_{Q(x_0, a)}$. Therefore any minimizing sequence (ρ_k) for the left hand side of (4.3) is bounded in $\mathcal{M}_+(\overline{\Omega})$ hence weakly* relatively compact. By the assertion iii) of Proposition 3.1, we will be able to conclude the convergence of infima in (4.3) if we can show that

$$(4.4) \quad F_{\varepsilon_k} + G \xrightarrow{\Gamma} F + G \quad \text{being} \quad G(\rho) := -\lambda \rho(Q(x_0, a)).$$

In virtue of Theorem 3.3, we already know that $F_{\varepsilon_k} \xrightarrow{\Gamma} F$. Then it is enough to invoke the stability property of the assertion iv) of Proposition 3.1. However the functional G given above is not weak* continuous on $\mathcal{M}(\overline{\Omega})$ and therefore, we need to verify the less stringent requirements set out in (3.2). The first one is satisfied since $|G(\rho)| \leq \lambda \|\rho\|$. For the second one, we observe that, if $F(\rho) < +\infty$, then ρ is of the form $\rho = u \mathcal{L}^d \llcorner \Omega$ (see Proposition 2.5), hence $\rho(\partial Q(x_0, a)) = 0$ and every sequence $\rho_n \xrightarrow{*} \rho$ satisfies $\rho_n(Q(x_0, a)) \rightarrow \rho(Q(x_0, a)) = \int_{Q(x_0, a)} u dx$. This confirms the validity of (4.4), hence that of (4.3). As a result the equality (4.2) is proved and, as announced, this achieves Step 2 and the proof of Theorem 3.3. \square

Proof of Lemma 4.1. (i) Let $u_k \rightarrow u$ in $L^1(\Omega)$. Then $\rho_k = u_k \mathcal{L}^d \llcorner \Omega$ and $\rho = u \mathcal{L}^d \llcorner \Omega$ are such that $\rho_k \xrightarrow{*} \rho$ in $\mathcal{M}_+(\overline{\Omega})$. Since $F = \Gamma\text{-}\lim F_{\varepsilon_k}$ is weak* lower semicontinuous, we infer that

$$\liminf_{k \rightarrow \infty} J(\overline{u}_k) = \liminf_{k \rightarrow \infty} F(\rho_k) \geq F(\rho) = J(u).$$

To show that $J(0) = 0$, we consider as S_k a singleton $\{x_0\}$ so that the associated measure $\rho_k = (\varepsilon_k)^d \delta_{x_0}$ satisfies $\rho_k \xrightarrow{*} 0$ while $F_{\varepsilon_k}(\rho_k) = 0$.

(ii) By the definition of the Γ -limit, $F(\rho) < +\infty$ implies that ρ is a weak* limit of a sequence (ρ_k) in $\mathcal{M}_+(\overline{\Omega})$, hence of the form $\rho = u \mathcal{L}^d \llcorner \Omega$

with $u \geq 0$. Let now $u \in L^1(\Omega; \mathbb{R}_+)$ and A a Borel subset of Ω . We show first that $J(u 1_A) \leq J(u)$ if $\mathcal{L}^d(\partial A) = 0$. We may assume that $J(u) < +\infty$. Hence, there exists a family of subsets $S_k \subset \Omega$ such that $\rho_k = \mu_{S_k}^{\varepsilon_k}$ satisfies $\rho_k \xrightarrow{*} \rho := u \mathcal{L}^d \llcorner \Omega$ and $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$. If we let $S'_k = S_k \cap A$ and $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$, we have $\rho'_k \xrightarrow{*} u 1_A \mathcal{L}^d \llcorner \Omega$, since indeed the convergence $\rho_k \xrightarrow{*} \rho$ is tight while $\rho(\partial A) = 0$. Therefore, $F_{\varepsilon_k}(\rho'_k) \leq F_{\varepsilon_k}(\rho_k)$. By passing to the limit, as $k \rightarrow +\infty$, we deduce that

$$J(u 1_A) \leq F(\rho 1_A) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) = J(u).$$

To extend the inequality to any Borel subset A of Ω , it is enough to consider an approximating sequence (A_n) such that

$$\mathcal{L}^d(\partial A_n) = 0 \quad , \quad \mathcal{L}^d(A_n \Delta A) \rightarrow 0,$$

and then pass to the limit in the inequality $J(u) \geq J(u 1_{A_n})$ while letting $n \rightarrow +\infty$. Indeed, the conclusion will then follow from the lower semicontinuity of J with respect to the norm convergence in $L^1(\Omega)$. Now, to construct such a sequence (A_n) , we consider a compact subset $K_n \subset A$ and an open subset $\omega_n \supset A$ such that $\mathcal{L}^d(\omega_n \setminus K_n) \leq \frac{1}{n}$. For every n , we can choose a suitable $r_n > 0$ such that the enlarged open set $A_n = K_n + B(0, r_n)$ satisfies $\mathcal{L}^d(\partial A_n) = 0$ ³ while $A_n \subset \omega_n$. Then clearly $\mathcal{L}^d(A_n \Delta A) \leq \mathcal{L}^d(\omega_n \setminus K_n) \rightarrow 0$.

Let us now prove now the assertion (iii). In a first step, we assume that $\text{spt}(u) \cap \text{spt}(v) = \emptyset$ so that there exists open subsets $A \supset \text{spt}(u)$ and $B \supset \text{spt}(v)$ such that $\text{dist}(A, B) := \eta > 0$.

We begin by proving the inequality $J(u+v) \geq J(u) + J(v)$. Without loss of generality, we may assume that $J(u+v) < +\infty$ (hence u and v are non-negative). Then there exists a sequence of sets $S_k \subset \bar{\Omega}$ such that

$$\rho_k = \mu_{S_k}^{\varepsilon_k} \xrightarrow{*} (u+v) \mathcal{L}^d \llcorner \Omega \quad , \quad F_{\varepsilon_k}(\rho_k) \rightarrow J(u+v).$$

We write $S_k = S'_k \cup S''_k$ where $S'_k = S_k \cap A$ and $S''_k = S_k \cap B$ are disjoint. Then, we have $\rho_k = \rho'_k + \rho''_k$ where $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$ and $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$. Clearly $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$ while $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$. Therefore, by applying the Γ -lim inf inequality to ρ'_k and ρ''_k while taking into account the super-additivity property of F_{ε_k} (see (2.1)), we deduce that:

$$\begin{aligned} J(u+v) &= \lim_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\geq F(u \mathcal{L}^d \llcorner \Omega) + F(v \mathcal{L}^d \llcorner \Omega) = J(u) + J(v). \end{aligned}$$

To show the converse inequality $J(u+v) \leq J(u) + J(v)$, we assume without any loss of generality that $J(u) < +\infty$ and $J(v) < +\infty$. Then

³Here we use the fact that the function $\alpha_n(r) = \mathcal{L}^d(\{x \in \Omega : \text{dist}(x, K_n) > r\})$ is bounded monotone non increasing so that it is continuous except possibly on a finite or countable subset of \mathbb{R}_+ .

we consider recovering sequences $\rho'_k = \mu_{S'_k}^{\varepsilon_k}$ and $\rho''_k = \mu_{S''_k}^{\varepsilon_k}$ such that $\rho'_k \xrightarrow{*} u \mathcal{L}^d \llcorner \Omega$, $\rho''_k \xrightarrow{*} v \mathcal{L}^d \llcorner \Omega$ and $F_{\varepsilon_k}(\rho_k) \rightarrow J(u)$, $F_{\varepsilon_k}(\rho''_k) \rightarrow J(v)$. Up to dropping the elements of S'_k which are not in A and the elements of S''_k which are not in B , we may assume that $\text{spt}(\rho'_k) \subset A$ and $\text{spt}(\rho''_k) \subset B$. Indeed, removing these points will not affect the weak* convergence to $u \mathcal{L}^d \llcorner \Omega$ and $v \mathcal{L}^d \llcorner \Omega$ respectively while the total energy ξ_{ε_k} will not increase. Therefore, by exploiting the right hand inequality in (2.1) and since $\rho_k = \rho'_k + \rho''_k$ converge weakly* to $\rho = (u + v) \mathcal{L}^d \llcorner \Omega$, we are led to the following set of inequalities:

$$\begin{aligned} J(u + v) &\leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(\rho_k) \leq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho'_k) + \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(\rho''_k) \\ &\quad + \limsup_{k \rightarrow \infty} \frac{2\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d} \|u\|_{L^1(\Omega)} \|v\|_{L^1(\Omega)} \\ &\leq J(u) + J(v) + C \limsup_{k \rightarrow \infty} \frac{\ell_+(\eta/\varepsilon_k)}{\varepsilon_k^d}, \end{aligned}$$

where, in the second line, we used the tight convergence of ρ'_k, ρ''_k . The conclusion follows by noticing that ℓ_+ coincides with ℓ for large values where it is non-increasing. Thus the integrability condition (H3) implies that $r^d \ell(r) \rightarrow 0$ as $r \rightarrow +\infty$. The desired sub-additivity inequality follows.

In a second step, we remove the strict separation condition on the supports of u and v by simply assuming that the upper-level sets $A := \{u > 0\}$ and $B := \{v > 0\}$ satisfy $\mathcal{L}^d(A \cap B) = 0$ (which, for u, v non-negative, is equivalent to say that $uv = 0$ a.e.). To that aim, possibly after substituting A, B with non-intersecting Borel representatives, we consider increasing sequences of compact subsets $K'_n \subset A, K''_n \subset B$ such that:

$$\mathcal{L}^d(A \setminus K'_n) \rightarrow 0, \quad \mathcal{L}^d(B \setminus K''_n) \rightarrow 0, \quad K'_n \cap K''_n = \emptyset.$$

Then $u_n := u \mathbf{1}_{K'_n}, v_n := v \mathbf{1}_{K''_n}$ satisfy $J(u_n + v_n) = J(u_n) + J(v_n)$. In virtue of assertion (ii), we infer that, for every n :

$$J(u_n + v_n) \leq J(u) + J(v) \quad \text{and} \quad J(u_n) + J(v_n) \leq J(u + v).$$

In virtue of the lower semicontinuity of J , since $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L^1(\Omega)$, we deduce from above the inequalities $J(u + v) \leq J(u) + J(v)$ and $J(u) + J(v) \leq J(u + v)$, hence the desired additivity property. \square

REFERENCES

- [1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

- [2] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.
- [3] Ugo Bindini and Guy Bouchitté. Relaxed many-body optimal transport and related asymptotics *Adv. Calc. Var., in Press, 2022*.
- [4] Xavier Blanc and Mathieu Lewin. The crystallization conjecture: a review. *EMS Surv. Math. Sci.*, 2(2):225–306, 2015.
- [5] Guy Bouchitté, Chloé Jimenez, and Rajesh Mahadevan. Asymptotic analysis of a class of optimal location problems. *J. Math. Pures Appl. (9)*, 95(4):382–419, 2011.
- [6] Guy Bouchitté and Michel Valadier. Integral representation of convex functionals on a space of measures. *J. Funct. Anal.*, 80(2):398–420, 1988.
- [7] Andrea Braides et al. *Gamma-convergence for Beginners*, volume 22. Clarendon Press, 2002.
- [8] Giuseppe Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations*, volume 207 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.
- [9] Gustave Choquet. Diamètre transfini et comparaison de diverses capacités. *Séminaire Brelot-Choquet-Deny. Théorie du potentiel*, 3, 1958-1959. talk:4.
- [10] Codina Cotar and Mircea Petrache. Next-order asymptotic expansion for n-marginal optimal transport with coulomb and riesz costs. *Advances in Mathematics*, 344:137–233, 2019.
- [11] Gianni Dal Maso. *An introduction to Γ -convergence*, volume 8. Springer Science & Business Media, 2012.
- [12] Douglas Hardin, Edward B. Saff, and Oleksandr Vlasiiuk. Asymptotic properties of short-range interaction functionals, 2021.
- [13] Douglas P. Hardin, Thomas Leblé, Edward B. Saff, and Sylvia Serfaty. Large deviation principles for hypersingular Riesz gases. *Constr. Approx.*, 48(1):61–100, 2018.
- [14] Fumio Hiai. Representation of additive functionals on vector-valued normed Köthe spaces. *Kodai Math. J.*, 2(3):300–313, 1979.
- [15] Fumio Hiai and Hisaharu Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivariate Anal.*, 7(1):149–182, 1977.
- [16] Ulrich Krengel. *Ergodic theorems*, volume 6 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.
- [17] Mathieu Lewin. Coulomb and Riesz gases: the known and the unknown. *J. Math. Phys.*, 63(6):Paper No. 061101, 77, 2022.
- [18] Christian Licht and Gérard Michaille. Global-local subadditive ergodic theorems and application to homogenization in elasticity. *Annales Mathématiques Blaise Pascal*, 9(1):21–62, 2002.
- [19] Mircea Petrache and Sylvia Serfaty. Next order asymptotics and renormalized energy for Riesz interactions. *J. Inst. Math. Jussieu*, 16(3):501–569, 2017.
- [20] Sylvia Serfaty. *Coulomb gases and Ginzburg-Landau vortices*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2015.
- [21] Sylvia Serfaty. Systems of points with Coulomb interactions. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 935–977. World Sci. Publ., Hackensack, NJ, 2018.

MEAN FIELD THEORY FOR SHORT-RANGE INTERACTION FUNCTIONALS 29

IMATH, UNIVERSITÉ DE TOULON, BP 20132, 83957 LA GARDE CEDEX-
FRANCE

Email address: `bouchitte@univ-tln.fr`

DEPTO. DE MATEMÁTICA, FCFM, UNIV. DE CONCEPCIÓN, CONCEPCIÓN,
CHILE

Email address, Mahadevan: `rmahadevan@udec.cl`